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Tractable likelihood-based estimation of non-linear DSGE models

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This paper presents a simple and fast maximum likelihood estimation method for non-linear DSGE models that are solved using a second- (or higher-) order accurate approximation. The method requires that the number of observables equals the number of exogenous shocks. Exogenous innovations are extracted recursively by inverting the observation equation, which allows easy computation of the likelihood function.

1. Introduction

Dynamic Stochastic General Equilibrium (DSGE) models are the workhorses of modern macroeconomics. A large literature has empirically estimated *linearized* DSGE models using likelihood-based methods (e.g., Kim, 2000; Otrok, 2001; Ireland, 2004). Linearized models cannot capture the effect of big shocks, or the role of risk for economic behavior—*non-linear* model solutions are needed for studying these phenomena. This paper presents a simple and fast maximum likelihood estimation method for non-linear DSGE models that are solved using second- (or higher-) order Taylor approximations (e.g., Sims, 2000; Kollmann, 2002; Schmitt-Grohé and Uribe, 2004; Lombardo and Sutherland, 2007). Those approximations provide the most tractable and widely used non-linear solution technique for medium- and large-scale DSGE models (Kollmann et al. (2011)). Thus, it is important to develop efficient methods for taking higher-order approximated models to data.

The *estimation* method discussed here requires that the number of observed variables (used for estimation) equals the number of exogenous shocks in the DSGE model. Exogenous innovations are extracted recursively by inverting the observation equation, which allows easy computation of the sample likelihood. A challenge for this approach is that, in higher-order approximated models, the decision rule (solution) for endogenous variables depends on powers of exogenous innovations—multiple exogenous innovations are thus consistent with the data. To overcome this problem, I posit a *modified* higher-order decision rule in which *powers* of exogenous innovations are replaced by their unconditional expected value. This allows straightforward observation equation inversion. A numerical example suggests that the estimation method here provides accurate parameter estimates, even for models with strong curvature and big shocks.[1]

Other likelihood-based estimation methods for non-linear DSGE models use particle filters or deterministic filters to infer exogenous shocks. The estimation method here is much simpler and faster; it can thus be applied to larger models.[2]

2. Model and estimation method

Standard DSGE models can be expressed as:

$$E_t G(\Omega_{t+1}, \Omega_t, \varepsilon_{t+1}) = 0, \tag{1}$$

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[1] Observation equation inversion is an intuitive and popular statistical technique (e.g., Guerrieri and Iacoviello, 2014; Deák et al., 2015), but has so far not been used to estimate higher-order approximated DSGE models. The paper here shows how this can be accomplished.

[2] Particle filters (PFs) use Monte Carlo methods to infer latent states (An and Schorfheide, 2007), and are thus computationally slow. Deterministic filters (DFs) are much faster than PFs, as DFs do not use Monte Carlos; instead updating rules akin to the standard Kalman filter are employed; this requires computation of conditional moments of the state vector (e.g., Kollmann, 2015). The method here is fastest as it does not involve computing moments of states. In contrast to the method here, PFs and DFs can be used when there are less observables than shocks.

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where G: $R^{2n+m} \rightarrow R^n$ is a function, and Ω_t is an $n \times 1$ vector of endogenous and exogenous variables known at date t; $\varepsilon_{t+1} \sim N\left(0, \xi^2 \Sigma_\varepsilon\right)$ is an $m \times 1$ vector of innovations to exogenous variables; ξ is a scalar indexing shock size. The model solution is a 'decision rule' $\Omega_{t+1} = F\left(X_t, \varepsilon_{t+1}, \xi\right)$, where X_t is a vector of state variables (predetermined endogenous variables and exogenous variables), i.e. $X_t = \Lambda\Omega_t$, where Λ is a matrix that picks the state variables among the elements of Ω_t. The decision rule has to satisfy $E_t G\left(F\left(\Lambda\Omega_t, \varepsilon_{t+1}, \xi\right), \Omega_t, \varepsilon_{t+1}\right) = 0 \quad \forall \Omega_t$. Following Sims (2000) and Schmitt-Grohé and Uribe (2004), this paper focuses on second-order accurate model solutions, namely on second-order Taylor approximations of the decision rule around a deterministic steady state, i.e. around $\xi = 0$ and a vector Ω such that $0 = G(\Omega, \Omega, 0)$.

The paper presents an estimation method for *second*-order approximated models. It is straightforward to extend the estimation method to models that are approximated to a *higher* order—see Appendix A (Not-for-Publication).

Second-order accurate solution and pruning

Let $\omega_t \equiv \Omega_t - \Omega, x_t \equiv X_t - X$ (with $X = \Lambda\Omega$) denote deviations of Ω_t, X_t from steady state. The second-order accurate model solution has the form

$$\omega_{t+1} = F_0\xi^2 + F_1x_t + F_2\varepsilon_{t+1} + F_{11}x_t \otimes x_t + F_{12}x_t \otimes \varepsilon_{t+1} + F_{22}\varepsilon_{t+1} \otimes \varepsilon_{t+1}, x_t = \Lambda\omega_t, \tag{2}$$

where $\otimes$ denotes the Kronecker product. $F_0, F_1, F_2, F_{11}, F_{12}, F_{22}$ are vectors/matrices that are functions of structural parameters. The first-order accurate (linearized) model solution is:

$$\omega_{t+1}^{(1)} = F_1x_t^{(1)} + F_2\varepsilon_{t+1}, \text{ with } x_t^{(1)} = \Lambda\omega_t^{(1)}. \tag{3}$$

The superscript $^{(1)}$ denotes variables solved to first-order accuracy. I assume that the eigenvalues of F_1 are strictly inside the unit circle, i.e. that the linearized model is stable.

I use the 'pruning' scheme of Kim et al. (2008), under which $x_t \otimes x_t$ and $x_t \otimes \varepsilon_{t+1}$ are replaced by $x_t^{(1)} \otimes x_t^{(1)}$ and $x_t^{(1)} \otimes \varepsilon_{t+1}$, respectively, in (2):

$$\omega_{t+1} = F_0\xi^2 + F_1x_t + F_2\varepsilon_{t+1} + F_{11}x_t^{(1)} \otimes x_t^{(1)} + F_{12}x_t^{(1)} \otimes \varepsilon_{t+1} + F_{22}\varepsilon_{t+1} \otimes \varepsilon_{t+1}. \tag{4}$$

Note that $x_t \otimes x_t = x_t^{(1)} \otimes x_t^{(1)}$ and $x_t \otimes \varepsilon_{t+1} = x_t^{(1)} \otimes \varepsilon_{t+1}$ hold, up to second-order accuracy. Thus, (4) is a valid second-order accurate solution. The justification for pruning is that (2) has spurious steady states (not present in the original model); some of those steady states mark transitions to unstable behavior. Large shocks can thus trigger explosive trajectories. Pruning eliminates this problem. Stability of the first-order solution (3) ensures that the pruned second-order solution is stable. Pruning is thus essential for applied work based on second-order approximated models.

The estimation method below uses data on ω_{t+1} to extract the exogenous innovation ε_{t+1}. As ω_{t+1} depends on squares of ε_{t+1}, multiple innovations are consistent with given data. To allow observation equation inversion, I replace the term $\varepsilon_{t+1} \otimes \varepsilon_{t+1}$ in (4) by its expected value $E(\varepsilon_{t+1} \otimes \varepsilon_{t+1})$.[3] This produces the '*modified*' decision rule

$$\omega_{t+1} = F_0\xi^2 + F_1x_t + F_2\varepsilon_{t+1} + F_{11}x_t^{(1)} \otimes x_t^{(1)} + F_{12}x_t^{(1)} \otimes \varepsilon_{t+1} + F_{22}E(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \tag{5}$$

that is *linear* in ε_{t+1}, but non-linear in lagged state variables. The subsequent discussion assumes that (5) is the **true** data generating process (DGP).[4]

Observation equation inversion

The estimation method here requires that the number of observables equals the number of exogenous innovations, m. Assume that the econometrician observes a vector z_{t+1} comprising m elements of the vector ω_{t+1}. Thus, the observation equation is $z_{t+1} = Q\omega_{t+1}$, where Q is an $m \times n$ matrix. Substituting (5) into the observation equation gives $z_{t+1} = \gamma_t + \lambda_t\varepsilon_{t+1}$, where $\gamma_t \equiv Q \cdot \left(F_0\xi^2 + F_1x_t + F_{11}x_t^{(1)} \otimes x_t^{(1)} + F_{22}E\left(\varepsilon_{t+1} \otimes \varepsilon_{t+1}\right)\right)$ and λ_t is an $m \times m$ matrix such that $\lambda_t\varepsilon_{t+1} \equiv Q \cdot \left(F_2\varepsilon_{t+1} + F_{12}x_t^{(1)} \otimes \varepsilon_{t+1}\right)$. Provided λ_t is non-singular, we thus have:

$$\varepsilon_{t+1} = \lambda_t^{-1}\left(z_{t+1} - \gamma_t\right). \tag{6}$$

Sample likelihood

Given the initial states $x_0^{(1)}, x_0$ and data $\{z_t\}_{t=1}^T$ one can recursively extract the exogenous innovations $\{\varepsilon_t\}_{t=1}^T$ using (3), (5) and (6). The log likelihood of the data (conditional on $x_0^{(1)}, x_0$) is:

$$\ln L\left(\{z_t\}_{t=1}^T | x_0^{(1)}, x_0\right) = -\frac{mT}{2}\ln(2\pi) - \frac{T}{2}\ln|\xi^2\Sigma_\varepsilon| - \frac{1}{2}\sum_{t=1}^T\left\{\varepsilon_t'\left(\xi^2\Sigma_\varepsilon\right)^{-1}\varepsilon_t - \ln|\lambda_{t-1}|\right\}. \tag{7}$$

Structural model parameters (and initial states) can be estimated by maximizing this function.

3. Illustration

The method is tested for a basic Real Business Cycle (RBC) model. Assume a representative household who maximizes date t lifetime utility V_t given by

$$V_t = \frac{1}{1-\sigma}C_t^{1-\sigma} - \frac{1}{1+\eta}\psi_tN_t^{1+\eta} + \lambda_t\beta E_tV_{t+1},$$

where C_t and N_t are consumption and hours worked. $\sigma, \eta > 0$ are risk aversion and the inverse of labor supply elasticity. $\lambda_t\beta$ is the subjective discount factor between t and $t+1$. $\psi_t, \lambda_t > 0$ are preference shocks. The resource/technology constraints are

$$C_t + I_t + G_t = Y_t, Y_t = \theta_tK_t^\alpha N_t^{1-\alpha}, K_{t+1} = (1-\delta)K_t + I_t, 0 < \alpha, \delta < 1.$$

$Y_t, K_t, I_t, G_t, \theta_t$ denote GDP, capital, investment, and exogenous government purchases and productivity. The exogenous variables follow

$$\ln(\theta_t) = \rho_\theta\ln(\theta_{t-1}) + \varepsilon_{\theta,t}, \ln(G_t/G) = \rho_G\ln(G_{t-1}/G) + \varepsilon_{G,t},$$
$$\ln(\psi_t) = \rho_\psi\ln(\psi_{t-1}) + \varepsilon_{\psi,t}, \ln(\lambda_t) = \rho_\lambda\ln(\lambda_{t-1}) + \varepsilon_{\lambda,t},$$

where $\varepsilon_{\theta,t}, \varepsilon_{G,t}, \varepsilon_{\psi,t}, \varepsilon_{\lambda,t}$ are normal white noises with standard deviations $s_\theta, s_G, s_\psi, s_\lambda$. As conventional in (quarterly) models, I set $\beta = 0.99, \eta = 0.25, \alpha = 0.3, \delta = 0.025, \rho_\theta = \rho_G = \rho_\psi = \rho_\lambda = 0.99$. The steady state government purchases/GDP ratio is $G/Y = 0.2$. Risk aversion is set at a high value, $\sigma = 10$, to generate strong curvature and permit non-negligible differences between first- and second-order model approximations (see Appendix A).

I normalize $\xi = 1$. One model variant assumes standard deviations $s_\theta = s_G = s_\psi = 1\%, s_\lambda = 0.025\%$ ('small shocks' variant) as in typical RBC models. The relative size of the shocks ensures that each shock accounts for a non-negligible share of the variance of GDP (see Appendix A). I also consider a 'big shocks' variant in which shocks are 5 times larger: $s_\theta = s_G = s_\psi = 5\%, s_\lambda = 0.125\%$. I solve the model using DYNARE.

[3] I thank Chris Sims for suggesting this approach. Dropping the term $F_{22}\varepsilon_{t+1} \otimes \varepsilon_{t+1}$ from (4) also permits observation equation inversion and produces very similar estimation results.

[4] For the illustrative DSGE model below, (4) and (5) are virtually indistinguishable: feeding the same sequence of innovations $\{\varepsilon\}$ into (4) and (5) produces almost identical time series $\{\omega\}$. Using (5) to extract the exogenous innovations (see below) from the time series generated by (4) and by (5) yields very similar parameter estimates. Thus, even if the true DGP is (4), one obtains reliable estimates by *positing* (5) for observation equation inversion.

Table 1
Monte Carlo: parameter estimates for second-order approximated RBC model.

	Model variant with 'small shocks'			Model variant with 'big shocks'		
Parameter	(1) Median	(2) Mean	(3) Std	(4) Median	(5) Mean	(6) Std
σ	10.18	12.01	5.11	10.08	10.29	1.65
η	0.27	0.59	0.79	0.22	0.32	0.26
ρ_θ	0.99	0.99	0.002	0.99	0.99	0.002
ρ_G	0.99	0.98	0.02	0.98	0.98	0.01
ρ_ψ	0.99	0.99	0.01	0.99	0.99	0.01
ρ_λ	0.97	0.93	0.09	0.99	0.96	0.07
$s_\theta(\%)$	1.00	1.00	0.07	4.94	4.99	0.45
$s_G(\%)$	0.99	1.00	0.08	5.04	5.09	0.48
$s_\psi(\%)$	1.02	1.28	0.73	4.99	5.16	1.17
$s_\lambda(\%)$	0.027	0.059	0.069	0.12	0.16	0.13

Note: The Table summarizes parameters estimates across 30 simulation runs ($T = 100$). Cols. labeled 'Median', 'Mean' and 'Std' report the median, mean and standard deviation of estimated parameters (see left-most column) across the 30 runs. Cols. labeled (1)-(3) [(4)-(6)]: 'small shocks' ['big shocks'] model variant. The *true* parameter values are: $\sigma = 10, \eta = 0.25, \rho_\theta = \rho_G = \rho_\psi = \rho_\lambda = 0.99$. True standard deviations of exogenous innovations in 'small shocks' variant: $s_\theta = s_G = s_\psi = 1\%,\ s_\lambda = 0.025\%$. 'Big shocks' variant: $s_\theta = s_G = s_\psi = 5\%$, $\sigma_\lambda = 0.125\%$.

For each model variant, I generated 30 simulation runs of 100 periods.[5] For each run, I estimated 10 model parameters by maximizing the likelihood function (7): the risk aversion coefficient (σ) , labor supply parameter (η) and autocorrelations and standard deviations of the exogenous variables. As the model has four exogenous shocks, four observables are needed for estimation. GDP, consumption, investment and hours worked are used as observables.

In computing the sample likelihood, I *assume* that initial states $x_0^{(1)}$, x_0 equal their unconditional mean. Although true initial states (in a given sample) differ from the assumed initial states, the recursively extracted exogenous innovations converge to the true innovations after a few periods. I thus use the first 10 periods of each simulation run as a training sample (the first 10 periods are dropped from the likelihood function).

One evaluation of the likelihood takes merely 0.014 seconds on a personal computer (Intel i7-7700K processor). This allows rapid maximization of the likelihood.

Table 1 reports the median, mean and standard deviation of the estimated model parameters across the 30 simulation runs, for the 'small shocks' model variant (Columns (1)–(3)) and for the 'big shocks' variant (Cols. (4)–(6)). Most model parameters are tightly estimated: the median and mean parameter estimates (across runs) are close to true parameter values, and the standard deviations of the parameter estimates are generally small.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.econlet.2017.08.027.

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[5] To eliminate the influence of initial conditions, the model was simulated over 5100 periods; the first 5000 periods were discarded.