Appendix: Supplementary material for

'Tractable Likelihood-Based Estimation of Non-Linear DSGE Models'

Economics Letters (available online 6 September 2017)
http://dx.doi.org/10.1016/j.econlet.2017.08.027

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6 September 2017

Part A. of this Appendix provides supplementary information for the second-order approximated RBC model discussed in the paper published in Economics Letters (Kollmann (2017)).

Part B. The technique described in Kollmann (2017) can also be used for likelihood estimation of DSGE models that are approximated to an order that is higher than the second order. Part B of this Appendix shows how *third*-order approximated DSGE models can be estimated using observation equation inversion.

A. Supplementary information for the second-order approximated RBC model considered in Kollmann (2017)

● Comparison between decision rule (4) and modified decision rule (5)

Table a1 documents that the decision rule (4) and the modified decision rule (5) discussed in Kollmann (2017) are (essentially) indistinguishable. An identical sequence of random exogenous innovations of length T=500,000 was fed into (4) and into (5). Table a1 shows that the resulting time series of endogenous variables are almost perfectly correlated across (4) and (5), and that they have (essentially) the same standard deviation. This holds both for levels and for first differences of logged simulated endogenous variables.

● Standard deviations of first- and second-order approximated models

Table a2 reports predicted standard deviations of first- and second-order approximated variables (log levels and log first differences). The Table documents that each of the four types of exogenous shocks accounts for a sizable share of the variance of GDP (see Panel (a), Col. (1)). In the 'small shocks' model variant, the first- and second-order approximated models produce almost identical standard deviations of endogenous variables (see Panel (a)). In the 'big shocks' model variant, by contrast, the second-order approximated variables are more volatile than the first-order approximated variables; this is, especially, the case for GDP, investment and hours worked (see Panel (b)).

Table a1. Second-order approximated RBC model: correlations across time series generated by decision rule (4) [ω] and time series generated by the 'modified' decision rule (5) [ω^{mod}]

	Y	*C*	*I*	*N*	*K*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with 'small shocks' (σ_θ=σ_G=σ_ψ=1%, σ_λ=0.025%)					
Correlations between ω and ω^{mod}					
Levels	1.0000	1.0000	1.0000	1.0000	1.0000
First differences	1.0000	1.0000	0.9999	1.0000	1.0000
Relative standard deviations: std(ω)/std(ω^{mod})					
Levels	1.0000	1.0000	1.0000	1.0000	1.0000
First differences	1.0000	1.0000	1.0000	1.0000	1.0000
Relative standard deviation of difference between decision rules: std(ω - ω^{mod})/std(ω^{mod})					
Levels	0.0006	0.0001	0.0030	0.0003	0.0001
First differences	0.0039	0.0012	0.0168	0.0033	0.0020
(b) Model variant with 'big shocks' (σ_θ=σ_G=σ_ψ=5%, σ_λ=0.125%)					
Correlations between ω and ω^{mod}					
Levels	1.0000	1.0000	0.9999	1.0000	1.0000
First differences	0.9998	1.0000	0.9967	0.9999	0.9999
Relative standard deviations: std(ω)/std(ω^{mod})					
Levels	1.0000	1.0000	0.9999	1.0000	1.0000
First differences	1.0000	1.0000	0.9994	1.0000	1.0000
Relative standard deviation of difference between decision rules: std(ω - ω^{mod})/std(ω^{mod})					
Levels	0.0027	0.0005	0.0147	0.0014	0.0007
First differences	0.0192	0.0061	0.0818	0.0164	0.0101

Note: Correlations of simulated time series (of variables listed above Cols. (1)-(5)) generated by the decision rule (4) and by the 'modified' decision rule (5) are reported, as well as the relative standard deviation of these two sets of time series. These statistics are reported for variables in log levels, and for variables in log first differences. *Y*: GDP; *C*: consumption; *I*: gross investment; *N*: hours worked; *K*: capital stock. Correlations greater than 0.99995 are reported as 1.0000. Reported statistics are based on one sequence of T=500,000 random exogenous innovations that was fed into (4) and (5).

Table a2. RBC model: predicted standard deviations (in %). Comparison between 1st order and 2nd order accurate model solutions

	Y	*C*	*I*	*N*	*K*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with 'small shocks' $(\sigma_\theta=\sigma_G=\sigma_\psi=1\%, \sigma_\lambda=0.025\%)$					
Variables in levels					
1st order, all shocks	3.34	1.57	10.43	9.68	7.59
1st order, just θ shock	2.07	1.36	6.20	9.32	4.52
1st order, just G shock	1.66	0.08	1.50	1.97	1.08
1st order, just ψ shock	1.14	0.75	3.43	0.96	2.49
1st order, just λ shock	1.66	0.21	7.51	1.61	5.48
2nd order, all shocks	3.34	1.57	10.43	9.68	7.59
First-differenced variables					
1st order, all shocks	0.67	0.17	2.60	1.13	0.18
2nd order, all shocks	0.67	0.17	2.60	0.13	0.18
(b) Model variant with 'big shocks' $(\sigma_\theta=\sigma_G=\sigma_\psi=5\%, \sigma_\lambda=0.125\%)$					
Variables in levels					
1st order, all shocks	16.72	7.83	52.14	48.39	37.95
2nd order, all shocks	17.11	7.83	52.97	48.67	38.21
First-differenced variables					
1st order, all shocks	3.33	0.86	12.98	5.66	0.91
2nd order, all shocks	3.41	0.87	13.37	5.77	0.92

Note: Standard deviations (in %) of simulated variables (listed above Cols. (1)-(5)) are shown for the RBC model. Rows labeled '1st order' and '2nd order' show standard deviations predicted by the first- and second-order accurate model solutions, respectively. The statistics are reported for variables in log levels, and for variables in log first differences. *Y*: GDP; *C*: consumption; *I*: gross investment; *N*: hours worked; *K*: capital stock. All statistics are computed using one simulation run of 500,000 periods.

B. Tractable Likelihood-Based Estimation of Third-Order Approximated DSGE Models

The technique described in Kollmann (2017) can also be used for likelihood estimation of DSGE models that are approximated to an order that is higher than the second order. This is illustrated here for **third-order** approximated models.

The third-order accurate model solution of the DSGE model (1) is given by:

$$\omega_{t+1}=F_0\xi^2+(F_1+F_{1\xi}\xi^2)x_t+(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+F_{11}x_t\otimes x_t+F_{12}x_t\otimes\varepsilon_{t+1}+F_{22}\varepsilon_{t+1}\otimes\varepsilon_{t+1}+\ldots$$

$$F_{111}x_t\otimes x_t\otimes x_t+F_{112}x_t\otimes x_t\otimes\varepsilon_{t+1}+F_{122}x_t\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1}+F_{222}\varepsilon_{t+1}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1},\ \text{with}\ x_t=\Lambda\omega_t. \qquad \text{(B.1)}$$

$F_{1\xi},F_{2\xi},F_{111},F_{112},F_{122},F_{222}$ are matrices that are functions of the structural model parameters ($(F_0,F_1,F_2,F_{11},F_{12},F_{22}$ are identical to the corresponding coefficients in the second-order accurate model solution; see (2) in Kollmann (2017)).

'Pruning' is also essential for applied work based on third-order approximated models--the 'un-pruned' system (B.1) can exhibit explosive dynamics, in response to big shocks (see discussion in Kollmann (2017)). To apply the logic of pruning to equation (B.1), note that the following conditions hold up to third-order accuracy:

$$\xi^2x_t=\xi^2x_t^{(1)},\quad x_t\otimes x_t=x_t^{(2)}\otimes x_t^{(1)}+x_t^{(2)}\otimes(x_t^{(2)}-x_t^{(1)}),\quad x_t\otimes\varepsilon_{t+1}=x_t^{(2)}\otimes\varepsilon_{t+1},$$

$$x_t\otimes x_t\otimes x_t=x_t^{(1)}\otimes x_t^{(1)}\otimes x_t^{(1)},\ x_t\otimes x_t\otimes\varepsilon_{t+1}=x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1},\ x_t\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1}=x_t^{(1)}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1},\ ^{1} \qquad \text{(B.2)}$$

where the superscript $^{(i)}$ denotes variables solved to i$^{\text{th}}$ accuracy and $x_t^{(i)}=\Lambda\omega_t^{(i)}$. The Dynare toolbox (Adjemian et al. (2014)) implements a pruned version of the third-order solution in which product terms in equation (B.1) are replaced by their third-order accurate equivalents stated in (B.2):

$$\omega_{t+1}=F_0\xi^2+F_1x_t+F_{1\xi}\xi^2x_t^{(1)}+(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+F_{11}\{x_t^{(2)}\otimes x_t^{(1)}+x_t^{(1)}\otimes(x_t^{(2)}-x_t^{(1)})\}+F_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+F_{22}\varepsilon_{t+1}\otimes\varepsilon_{t+1}+\ldots$$

$$F_{111}x_t^{(1)}\otimes x_t^{(1)}\otimes x_t^{(1)}+F_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}+F_{122}x_t^{(1)}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1}+F_{222}\varepsilon_{t+1}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1}. \qquad \text{(B.3)}$$

(This pruned third-order solution was also proposed by Kollmann (2004).) The dynamics of the first- and second-order approximated quantities is governed by (3) and (4) in Kollmann (2017), restated here for convenience:

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[1]For variable a_t we can write $a_t=a_t^{(1)}+R^{(2)}$ and $a_t=a_t^{(2)}+R^{(3)}$, where $R^{(n)}$ contains terms of order n or higher in deviations from the steady state. The product a_tb_t can thus be expressed as $a_tb_t=(a_t^{(1)}+a_t^{(2)}-a_t^{(1)}+R^{(3)})(b_t^{(1)}+b_t^{(2)}-b_t^{(1)}+R^{(3)})=a_t^{(1)}b_t^{(2)}+(a_t^{(2)}-a_t^{(1)})b_t^{(1)}+R^{(4)}$; hence, $(a_tb_t)^{(3)}=a_t^{(1)}b_t^{(2)}+(a_t^{(2)}-a_t^{(1)})b_t^{(1)}$. (Note that $a_t^{(2)}-a_t^{(1)}=R^{(2)}$, and hence $(a_t^{(2)}-a_t^{(1)})(b_t^{(2)}-b_t^{(1)})=R^{(4)}$.) The same logic shows that $(a_tb_tc_t)^{(3)}=a_t^{(1)}b_t^{(1)}c_t^{(1)}$.

$$\omega_{t+1}^{(1)}=F_1 x_t^{(1)}+F_2\varepsilon_{t+1},\ \ \omega_{t+1}^{(2)}=F_0\xi^2+F_1 x_t^{(2)}+F_2\varepsilon_{t+1}+F_{11}x_t^{(1)}\otimes x_t^{(1)}+F_{12}x_t^{(1)}\otimes\varepsilon_{t+1}+F_{22}\varepsilon_{t+1}\otimes\varepsilon_{t+1}. \qquad \text{(B.4)}$$

The moving average representation of the third-order pruned solution (B.3) depends on first-, second and third-order terms in exogenous innovations (ε), but not on higher-order terms. The third-order pruned system (B.3) is stationary if the first-order system is stationary.

To allow observation equation inversion, I replace squares and cubes of ε_{t+1} in (B.3) by their expected values. This gives the '*modified*' decision rule

$$\omega_{t+1}=F_0\xi^2+F_1x_t+F_{1\xi}\xi^2x_t^{(1)}+(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+F_{11}\{x_t^{(2)}\otimes x_t^{(1)}+x_t^{(1)}\otimes(x_t^{(2)}-x_t^{(1)})\}+F_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+F_{22}E(\varepsilon_{t+1}\otimes\varepsilon_{t+1})+...$$
$$F_{111}x_t^{(1)}\otimes x_t^{(1)}\otimes x_t^{(1)}+F_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}+F_{122}x_t^{(1)}\otimes E(\varepsilon_{t+1}\otimes\varepsilon_{t+1}). \qquad \text{(B.5)}$$

Note that $E(\varepsilon_{t+1}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1})=0$, because ε_{t+1} is normally distributed. The subsequent discussion assumes that (B.5) is the **true** data generating process.

Assume that the econometrician observes a vector z_{t+1} comprising m elements of the vector ω_{t+1} (recall that m is the number of exogenous innovations). Thus, the observation equation is $z_{t+1}=Q\cdot\omega_{t+1}$, where Q is an mxn selection matrix. Substitution of equation (B.5) into the observation equation gives $z_{t+1}=\gamma_t+\lambda_t\varepsilon_{t+1}$, where

$$\gamma_t\equiv Q\cdot[F_0\xi^2+F_1x_t+F_{1\xi}\xi^2x_t^{(1)}+F_{11}\{x_t^{(2)}\otimes x_t^{(1)}+x_t^{(1)}\otimes(x_t^{(2)}-x_t^{(1)})\}+F_{22}E(\varepsilon_{t+1}\otimes\varepsilon_{t+1})+F_{111}x_t^{(1)}\otimes x_t^{(1)}\otimes x_t^{(1)}+F_{122}x_t^{(1)}\otimes E(\varepsilon_{t+1}\otimes\varepsilon_{t+1})]$$

and λ_t is an $m\text{x}m$ matrix such that $\lambda_t\varepsilon_{t+1}\equiv Q\cdot[(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+F_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+F_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}]$. Provided λ_t is non-singular, we thus have:

$$\varepsilon_{t+1}=\lambda_t^{-1}(z_{t+1}-\gamma_t)\,. \qquad \text{(B.6)}$$

Given the initial states $x_0^{(1)},x_0^{(2)},x_0$ and data $\{z_t\}_{t=1}^T$ one can recursively extract the innovations $\{\varepsilon_t\}_{t=1}^T$ using (B.4),(B.5) and (B.6). The log likelihood of the data (conditional on $x_0^{(1)},x_0^{(2)},x_0$) is:

$$\ln L(\{z_t\}_{t=1}^T|\,x_0^{(1)},x_0^{(2)},x_0)=-\tfrac{mT}{2}\ln(2\pi)-\tfrac{T}{2}\ln|\xi^2\Sigma_\varepsilon|-\tfrac{1}{2}\sum\nolimits_{t=1}^T\{\varepsilon_t'\,(\xi^2\Sigma_\varepsilon)^{-1}\varepsilon_t-\ln|\lambda_{t-1}|\}\,. \qquad \text{(B.7)}$$

Structural model parameters (and the initial states) can be estimated by maximizing this function.

Illustration: RBC model, approximated to third-order

I compute a third-order approximation of the RBC model described in Kollmann (2017). Both the 'small shocks' variant of that model, and the 'big shocks' variant are considered. Table b1 documents that decision rule (B.3) and the modified decision rule (B.5) are (essentially) indistinguishable. An identical sequence of random exogenous innovations of length T=500,000 was fed into (B.3) and into (B.5). Table b1 shows that the resulting time series of endogenous variables are almost perfectly correlated across (B.3) and (B.5), and that they have (essentially) the same standard deviations. This holds both for levels and for first differences of logged simulated endogenous variables.

Table b2 reports predicted standard deviations of first-, second- and third-order approximated variables (log levels and log first differences). In the 'big shocks' RBC model variant, GDP, investment and capital are noticeably more volatile under a third-order approximation than under first- or second-order approximations (see Panel (b)).

Finally, I estimate the model parameters using simulated time series, by maximizing the likelihood function (B.7). As for the Monte Carlo described in Kollmann (2017), I generated 30 simulation runs of 100 periods each.[2] In computing the sample likelihood, I *assume* that the initial states $x_0^{(1)}, x_0^{(2)}, x_0$ equal their unconditional mean. The first 10 periods in each simulation run are used as a training sample. Table b3 reports the median, mean and standard deviation of the estimated model parameters across the 30 simulation runs, for the 'small shocks' model variant (Columns (1)-(3)) and for the 'big shocks' variant (Cols. (4)-(6)). As for the second-order accurate model discussed in Kollmann (2017), most model parameters are tightly estimated.

References

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[2]To eliminate the influence of initial conditions, the model was simulated over 5100 periods; the first 5000 periods were discarded.

Table b1. <u>Third-order</u> approximated RBC model: correlations across time series generated by decision rule (B.3) [ω] and time series generated by 'modified' decision rule (B.5) [ω^{mod}]

	Y	C	I	N	K
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with 'small shocks' (σ_θ=σ_G=σ_ψ=1%, σ_λ=0.025%)					
Correlations between ω and $\omega^{non-\text{mod}}$					
Levels	1.0000	1.0000	1.0000	1.0000	1.0000
First differences	1.0000	1.0000	0.9999	1.0000	1.0000
Relative standard deviations: std(ω)/std(ω^{mod})					
Levels	1.0000	1.0000	1.0000	1.0000	1.0000
First differences	1.0000	1.0000	0.9996	1.0000	1.0000
Relative standard deviation of difference between decision rules: std(ω - ω^{mod})/std(ω^{mod})					
Levels	0.0006	0.0001	0.0030	0.0003	0.0001
First differences	0.0039	0.0012	0.0169	0.0033	0.0020
(b) Model variant with 'big shocks' (σ_θ=σ_G=σ_ψ=5%, σ_λ=0.125%)					
Correlations between ω and $\omega^{non-\text{mod}}$					
Levels	1.0000	1.0000	0.9999	1.0000	1.0000
First differences	0.9998	1.0000	0.9961	0.9999	1.0000
Relative standard deviations: std(ω)/std(ω^{mod})					
Levels	1.0000	1.0000	0.9994	1.0000	1.0000
First differences	0.9998	1.0000	0.9910	1.0000	1.0000
Relative standard deviation of difference between decision rules: std(ω - ω^{mod})/std(ω^{mod})					
Levels	0.0024	0.0005	0.0155	0.0015	0.0007
First differences	0.0179	0.0065	0.0896	0.0168	0.0101

Note: Correlations of simulated time series (of variables listed above Cols. (1)-(5)) generated by decision rule (B.3) and by the 'modified' decision rule (B.5) are reported, as well as the relative standard deviation of these two sets of time series. The statistics are reported for log levels and for log first differences of endogenous variables. *Y*: GDP; *C*: consumption; *I*: gross investment; *N*: hours worked; *K*: capital stock. Correlations greater than 0.99995 are reported as 1.0000. Reported statistics are based on one simulation run of 500,000 periods.

Table b2. RBC model: predicted standard deviations (in %). Comparison between 1st order, 2nd order and 3rd order accurate model solutions

	Y	*C*	*I*	*N*	*K*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with 'small shocks' (σ_θ=σ_G=σ_ψ=1%, σ_λ=0.025%)					
Variables in levels					
1st order	3.34	1.57	10.43	9.68	7.59
2nd order	3.34	1.57	10.43	9.68	7.59
3rd order	3.36	1.57	10.48	9.67	7.62
First-differenced variables					
1st order	0.67	0.17	2.60	1.13	0.18
2nd order	0.67	0.17	2.60	1.13	0.18
3rd order	0.67	0.17	2.61	1.13	0.18
(b) Model variant with 'big shocks' (σ_θ=σ_G=σ_ψ=5%, σ_λ=0.125%)					
Variables in levels					
1st order	16.72	7.83	52.14	48.39	37.95
2nd order	17.11	7.83	52.97	48.67	38.21
3rd order	19.70	7.83	59.36	47.90	42.86
First-differenced variables					
1st order	3.33	0.86	12.98	5.66	0.91
2nd order	3.41	0.87	13.37	5.77	0.92
3rd order	3.80	0.85	14.46	5.87	0.96

Note: Standard deviations (in %) of simulated variables (listed above Cols. (1)-(5)) are shown for the RBC model. Rows labeled '1st order', '2nd order' and '3rd order' show standard deviations predicted by the first-, second- and third- order accurate model solutions, respectively. The statistics are reported for log levels and for log first differences of endogenous variables. *Y*: GDP; *C*: consumption; *I*: gross investment; *N*: hours worked; *K*: capital stock. All statistics are computed using one simulation run of 500,000 periods.

Table b3. Monte Carlo: parameter estimates for <u>third-order</u> approximated RBC model

Parameter	Model variant with 'small shocks'			Model variant with 'big shocks'		
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
	Median	**Mean**	**Std**	**Median**	**Mean**	**Std**
σ	10.67	11.17	2.74	10.95	12.42	3.89
η	0.31	0.43	0.47	0.45	0.69	0.69
ρ_θ	0.99	0.99	0.003	0.99	0.99	0.003
ρ_G	0.98	0.98	0.01	0.98	0.98	0.03
ρ_ψ	0.99	0.99	0.01	0.99	0.98	0.01
ρ_λ	0.96	0.95	0.05	0.98	0.96	0.05
s_θ (%)	0.99	1.00	0.07	4.97	4.97	0.34
s_G (%)	0.98	0.98	0.09	4.77	4.88	0.59
s_ψ (%)	0.97	1.13	0.40	5.34	6.86	3.24
s_λ (%)	0.035	0.042	0.025	0.18	0.19	0.09

Note: The Table summarizes parameters estimates across 30 simulation runs of 100 periods. Cols. labelled 'Median', 'Mean' and 'Std' report the median, mean and standard deviation of estimated parameters (listed in left-most column) across the 30 runs. Cols. labelled (1)-(3): 'small shocks' model variant. Cols. (4)-(6): 'big shocks' model variant.

The ***true*** parameter values are: σ=10, η=0.25, $\rho_\theta=\rho_G=\rho_\psi=\rho_\lambda=0.99$. True standard deviations of exogenous innovations in 'small shocks' model variant: $s_\theta=s_G=s_\psi=1\%$, $s_\lambda=0.025\%$. 'Big shocks' variant: $s_\theta=s_G=s_\psi=5\%$, $\sigma_\lambda=0.125\%$.