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Solving the incomplete market model with aggregate uncertainty using a perturbation method

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ABSTRACT

We use a perturbation method to solve the incomplete markets model with aggregate uncertainty described in den Haan et al. [Computational suite of models with heterogeneous agents: incomplete markets and model uncertainty. Journal of Economic Dynamics & Control, this issue]. To apply that method, we use a "barrier method" to replace the original problem with occasionally binding inequality constraints by one with only equality constraints. We replace the structure with a continuum of agents by a setting in which a single infinitesimal agent faces prices generated by a representative-agent economy. We also solve a model variant with a large (but finite) number of agents. Our perturbation-based method is much simpler and faster than other methods.

1. Introduction

This paper explains how a perturbation method can be applied to solve the incomplete markets model with aggregate uncertainty described in den Haan et al. (2009). We face two main challenges in applying existing perturbation algorithms to this model: how to deal with the occasionally binding non-negativity constraint for capital and with the continuum of agents. In Sections 2 and 3, we explain how we meet these challenges by using a barrier method, and by replacing the structure with a continuum of agents by a setting in which a single infinitesimal agent faces prices generated by a representative-agent economy; we also discuss a model variant with a large (but finite) number of agents.[1] Note that the simulation results reported in the comparison paper, den Haan (2009), are based on the representative-agent setup. Sections 4 and 5 summarize key properties of the model solution, including policy functions and Euler equation errors. Section 6 concludes the paper.

2. Two challenges for perturbation

2.1. Continuum of agents

The model is a production economy with a continuum of households of unit mass. Currently available "general purpose" computer programs for perturbation analysis are not designed to deal with an infinite number of

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[1] Kim et al. (2004) uses the same approach to solve a heterogeneous agent incomplete markets model with trade in bonds (instead of trade in physical capital).

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variables.[2] In order to apply the perturbation method to this model, we make use of the property that each infinitesimal agent is a price taker. We solve a representative agent version of the economy to generate a process for the wage rate and the rental rate of capital. That is, we express the wage and rental rates as functions of aggregate shocks and the aggregate capital stock only, while ignoring the wealth distribution. We then solve the decision problem of an infinitesimal agent who faces that process.[3]

The assumption that factor prices are generated by a representative-agent economy greatly simplifies the application of the perturbation method, but that assumption is not indispensable. We also consider a model variant in which the continuum of agents is replaced by a large (but finite) number of agents, and in which individual decisions and factor prices are jointly solved for. We find that, as the number of agents rises, individual policy functions in that N-agent model approach those in the "representative-agent" setup.

In closely related research on a heterogenous agent economy, Preston and Roca (2007) explicitly include the second moments of the wealth distribution as a perturbation variable, and solve the model using a second-order accurate perturbation method. Therefore, the wealth distribution in their solution is consistent with individual behavior. However, given the second-order nature of the wealth distribution terms, ignoring the wealth distribution would not affect the results when the model is approximated up to the first order, which is in fact the approach taken in most of this paper.[4]

In another closely related contribution, Reiter (2009) seeks to overcome the local nature of the perturbation method by combining that method with a projection method. Specifically, he solves a model variant with only idiosyncratic shocks by a projection method, and then perturbs its solution with respect to aggregate shocks (up to the first order in his application).

Compared to these contributions, our approach provides the simplest way to apply a perturbation method to the model with heterogeneous agents. The method here is also very fast, especially when compared to projection methods; for example, computing policy functions takes less than a second.[5]

2.2. Inequality constraint

The other challenge is how to deal with the non-negativity constraint for individual capital stocks. Perturbation methods cannot directly be applied to models with occasionally binding inequality constraints. One possible way to deal with this problem is to modify the utility function so that agents are penalized when capital holdings move close to the "barrier" of the zero bound. This "barrier method" (see Luenberger, 1973; Judd, 1998) converts the model with inequality constraints into an optimization problem with only equality constraints, which allows us to apply a standard perturbation method.[6]

Specifically, we consider the following modified utility function:

$$U(c_t^i, k_t^i) = \frac{(c_t^i)^{1-\gamma} - 1}{1-\gamma} + \phi\left[\log\left(\frac{k_t^i}{\bar{k}}\right) - \frac{k_t^i - \bar{k}}{\bar{k}}\right], \tag{1}$$

where c_t^i, k_t^i are an individual household's consumption in period t, and her capital stock at the beginning of that period, respectively. $\phi > 0$ is a coefficient (referred to as a "barrier parameter"), and $\bar{k}$ is the value of the individual capital stock in the (deterministic) steady state of the economy. Due to the term $\log k_t^i$ in the (modified) utility function, the marginal utility of holding capital goes to infinity when k_t^i goes towards zero. This specification ensures that the steady state is independent of ϕ. (Note that this modified utility function penalizes not only capital holdings below the steady state but also capital holdings above the steady state.)

We next describe the model equations used for our computational approach.

3. The model

The budget constraint of an individual household is given by

$$c_t^i + k_{t+1}^i = r_t k_t^i + (1-\tau_t) w_t \bar{l} \varepsilon_t^i + \mu w_t (1-\varepsilon_t^i) + (1-\delta) k_t^i, \tag{2}$$

where $\varepsilon_t^i \in \{0,1\}$ is the employment status of the individual; $\bar{l}$ is the time endowment of the household (set at 1/0.9); r_t and w_t are the rental rate of capital and the wage rate, respectively; τ_t is the labor tax rate; μw_t is the unemployment benefit;

[2] See algorithms such as gensys, gensys2, Dynare, PerturbationAIM, and Judd-Jin's PSD. For a detailed comparison of these methods, see Kim et al. (2008).

[3] When solving a heterogeneous-agent model, Krusell and Smith (1998) also assume that households believe that they face a simple stochastic process for aggregate capital and its price. These authors impose the consistency condition that the process for aggregate prices faced by households must be the same as the one that they would estimate from a simulation of the model. The representative-agent structure in this paper does not have such a consistency requirement.

[4] In the economy considered in Preston and Roca (2007), the distribution of capital does not significantly affect aggregate dynamics.

[5] Participating algorithms based on projection methods take between 7 and 2739 min to solve the model. See Table 3 in den Haan's (2009) comparison paper.

[6] Preston and Roca (2007) also use a barrier method.

and δ is the depreciation rate of capital. The household's intertemporal Euler equation is

$$(c_t^i)^{-\gamma} = E_t\beta(c_{t+1}^i)^{-\gamma}(r_{t+1}+1-\delta)+\beta\phi\left(\frac{1}{k_{t+1}^i}-\frac{1}{\bar{k}}\right), \quad (3)$$

where β is the subjective discount factor.

The equilibrium rental rate and the wage rate are given by

$$r_t = \alpha a_t\left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha-1}, \quad w_t = (1-\alpha)a_t\left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha}, \quad (4)$$

where a_t, K_t, and L_t are TFP, the aggregate capital stock, and the aggregate labor supply, respectively.

The first model variant that we consider, referred to as the "representative agent setup," solves the rental and wage rates from a representative agent economy; that variant assumes that r_t satisfies the following Euler equation (of a hypothetical representative household):

$$(C_t)^{-\gamma} = E_t\beta(C_{t+1})^{-\gamma}(r_{t+1}+1-\delta), \quad (5)$$

where C_t is aggregate consumption. The aggregate resource constraint is

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t, \quad (6)$$

where Y_t is aggregate output. The aggregate production function is $Y_t = a_t(K_t)^{\alpha}(\bar{l}L_t)^{1-\alpha}$. We assume that L_t is given by

$$(L_t - \bar{L}) = 3(a_t - 1), \quad (7)$$

where $\bar{L}$ is the unconditional mean of total employment; this equation guarantees that, in the economy with aggregate uncertainty, the unemployment rate is 4% in a state with high TFP ($a_t = 1.01$) and 10% in a state with low TFP ($a_t = 0.99$), as stipulated in the description of the model in den Haan et al. (2009). The labor tax rate obeys $\tau_t = \mu(1-L_t)/(\bar{l}L_t)$.

The equilibrium conditions of the "representative-agent setup" consist of Eqs. (2)–(7).

The second model variant, the "N-agent model," assumes $N>1$ households indexed by $i = 1,\dots,N$. The equilibrium conditions of that variant are given by Eqs. (2) and (3) for $i = 1,\dots,N$, by (6) and (4), and the aggregation conditions $C_t = (1/N)\sum_i c_t^i$, $K_t = (1/N)\sum_i k_t^i$, $L_t = (1/N)\sum_i \varepsilon_t^i$.

We approximate both model variants around steady states values of aggregate and individual capital stocks and consumption that are given by

$$\bar{K} = \bar{k} = \bar{l}\bar{\varepsilon}\left(\frac{\delta+(1-\beta)/\beta}{\alpha}\right)^{(1/(\alpha-1))}, \quad (8)$$

$$\bar{C} = \bar{c} = \bar{k}^{\alpha}(\bar{l}\bar{\varepsilon})^{1-\alpha} - \delta\bar{k}, \quad (9)$$

where $\bar{\varepsilon}$ is the (exogenously given) mean employment rate. Throughout the paper, we denote steady state values by an upper bar.

3.1. Calibration of continuous-state shock processes

Although the perturbation method does not generically require a continuous distribution for the shocks, the algorithms that we use (see below) require continuous-state autoregressive forcing variables. Therefore, we construct continuous approximations of the discrete-state Markov processes described in den Haan et al. (2009). We calibrate the continuous processes by matching one-period-ahead conditional expected values and standard deviations of individual employment and of TFP to the corresponding moments of the original discrete-state processes (details available on request).

In the *economy without aggregate uncertainty*, this moment-matching approach yields the following continuous process for individual employment:

$$\varepsilon_{t+1}^i = 0.4 + .55555\varepsilon_t^i + (0.48989 - 0.28381\varepsilon_t^i)\eta_{t+1}^i, \quad (10)$$

where η_{t+1}^i is a white noise that has unit variance and is independent across agents (thus $\int \eta_{t+1}^i\, di = 0$).[7]

In the *economy with aggregate uncertainty*, the same moment-matching approach yields the following process for TFP:

$$a_{t+1} = 0.25 + 0.75a_t + 0.00661\eta_{t+1}^a, \quad (11)$$

[7] Under the discrete-state process ε_t^i only takes values 0 or 1, and $E(\varepsilon_{t+1}^i|\varepsilon_t^i=0)=0.4$, $E(\varepsilon_{t+1}^i|\varepsilon_t^i=1)=0.9555$, $Std(\varepsilon_{t+1}^i|\varepsilon_t^i=0)=0.4898$, $Std(\varepsilon_{t+1}^i|\varepsilon_t^i=1)=0.2060$. The process (10) reproduces those four moments; e.g. (10) implies $E[\varepsilon_{t+1}^i|\varepsilon_t^i]=0.4+.5555\varepsilon_t^i$; thus $E[\varepsilon_{t+1}^i|\varepsilon_t^i=1]=0.4+.5555=0.9555$, etc.

where η^a_{t+1} is a white noise with unit variance. Eq. (7) implies the following linear relation between a_t and the cross-agent mean employment rate at date t, here denoted by $\overline{\varepsilon_t}$: $\overline{\varepsilon_t} = -2.07 + 3a_t$.[8] In the *economy with aggregate uncertainty*, we assume that agent i's employment follows a process of the form $\varepsilon^i_{t+1} - \overline{\varepsilon_{t+1}} = \{\gamma_1 + \gamma_2 a_{t+1} + \gamma_3 a_t + \gamma_4(\varepsilon^i_t - \overline{\varepsilon_t})\}(\varepsilon^i_t - \overline{\varepsilon_t}) + \{\gamma_5 + \gamma_6 a_{t+1} + \gamma_7 a_t + \gamma_8(\varepsilon^i_t - \overline{\varepsilon_t})\}\eta^i_{t+1}$, where η^i_{t+1} is a white noise with unit variance; η^i_{t+1} is independent across agents and independent of η^a_{t+1}. This specification guarantees that the average employment rate in period $t+1$ equals $\overline{\varepsilon_{t+1}} : \int \varepsilon^i_{t+1} di = \overline{\varepsilon_{t+1}}$ if $\int \varepsilon^i_t di = \overline{\varepsilon_t}$. We select the coefficients $\gamma_1, \ldots, \gamma_8$ that best match the one-period ahead conditional moments $E[\varepsilon^i_{t+1}|a_{t+1}, a_t, \varepsilon^i_t]$, $Std[\varepsilon^i_{t+1}|a_{t+1}, a_t, \varepsilon^i_t]$ of the discrete-state process, for $a_t, a_{t+1} \in \{0.99, 1.01\}$, $\varepsilon^i_t \in \{0, 1\}$.[9] This gives

$$\varepsilon^i_{t+1} - \overline{\varepsilon_{t+1}} = \{13.0158 - 17.4167a_{t+1} + 4.8438a_t\}(\varepsilon^i_t - \overline{\varepsilon_t}) + \{2.5159 - 2.4029a_{t+1} + 0.0953a_t - 0.2672(\varepsilon^i_t - \overline{\varepsilon_t})\}\eta^i_{t+1}. \quad (12)$$

3.2. Numerical algorithm

We solve the model variants up to first order, using the Matlab code `gensys.m` (Sims, 2001). (Note that all second-order terms in the continuous shock processes described in the previous section disappear when a first-order accurate model approximation is computed.) In addition, we experiment with a second-order accurate solution, using the Matlab code `gensys2.m` (Kim et al., 2008).[10]

3.3. Calculating policy functions

We use levels of variables as perturbation variables.[11] In what follows, $dz \equiv z - \bar{z}$ denotes the deviation of a variable z from its steady state value $\bar{z}$. For the "representative-agent setup," the program generates a solution of the following form:

$$dv^i_t = f(dy^i_{t-1,}[\eta^i_t, \eta^a_t]), \quad (13)$$

where $v^i_t \equiv [c^i_t, \varepsilon^i_t, k^i_{t+1}; C_t, L_t, K_{t+1}; a_t]$, $y^i_{t-1} \equiv [\varepsilon^i_{t-1}, k^i_t, K_t, a_{t-1}]$.

The model requires a policy function that expresses v^i_t as a function of $z^i_t \equiv [\varepsilon^i_t, k^i_t, K_t, a_t]$, i.e. of individual and aggregate capital stocks at the beginning of period t, and of employment and TFP at t. Such a policy function can be obtained by using the shock processes (10)–(12) to express $[\eta^i_t, \eta^a_t]$ as functions of $[d\varepsilon^i_t, d\varepsilon^i_{t-1}, da_t, da_{t-1}]$. Substitution of the resulting expressions into (13) gives $dv^i_t = \tilde{f}(dz^i_{t,}[d\varepsilon^i_{t-1}, da_{t-1}])$. Of course, the derivatives of $\tilde{f}$ with respect to $[d\varepsilon^i_{t-1}, da_{t-1}]$ are zero. Hence, the policy function is given by

$$dv^i_t = F(dz^i_t) \equiv \tilde{f}(dz^i_{t,}[0, 0]). \quad (14)$$

3.3.1. Adjusting the policy function to ensure non-negativity of the capital stock

If one solved the optimization problem with the modified utility function (1) in an exact nonlinear way, the solution for individual capital would never hit the zero lower-bound, and that even for arbitrarily small values of ϕ. In our approximate solution, however, a sizable value of ϕ is required to ensure that agents never reach the boundary; for small values of ϕ, agents can violate the inequality constraint as the perturbation method generates a smooth solution that includes the negative range for end-of-period capital. When the inequality constraint is violated, we force the agents to stay on the boundary by using the following strategy. Let k^{i*}_{t+1}, c^{i*}_t denote the date t capital and consumption choices generated by the perturbation solution. We impose the non-negativity constraint $k^i_{t+1} \geq 0$ by replacing k^{i*}_{t+1} and c^{i*}_t by $k^i_{t+1} = \max(0, k^{i*}_{t+1})$ and $c^i_t = c^{i*}_t + k^{i*}_{t+1} - k^i_{t+1}$. In other words, when the perturbation solution generates negative capital $k^{i*}_{t+1} < 0$, we replace k^{i*}_{t+1} by $k^i_{t+1} = 0$, and we use the budget constraint to solve for consumption.

4. Results for the "representative-agent setup"

This section discusses selected quantitative results for the "representative-agent setup." We focus on the role of the barrier coefficient ϕ for model properties. Additional simulation results can be found in den Haan's (2009) comparison paper.

[8] Under the discrete-state process with aggregate uncertainty, $a_t \in \{0.99, 1.01\}$, and $\overline{\varepsilon_t}$ only takes the value 0.9 or 0.96 ($\overline{\varepsilon_t} = 0.9$ when $a_t = 0.99$, and $\overline{\varepsilon_t} = 0.96$ when $a_t = 1.01$). Our linear relation matches those values.

[9] Specifically, the selected $\gamma_1, \ldots, \gamma_8$ coefficients minimize the sum of squared differences between the conditional one-period ahead moments implied by the continuous approximating process and the conditional moments of the discrete-state process.

[10] These two programs are publicly available at http://sims.princeton.edu/yftp/gensys2. Robert Kollmann also contributed to the development of gensys2.

[11] It would not be suitable to use the log capital stock as a perturbation variable, as this would entail that the solution for capital is always strictly positive. In the given model, a non-negligible fraction of agents hold zero capital.

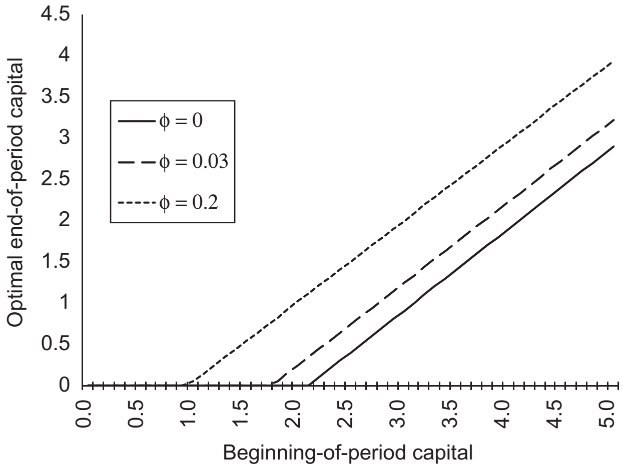

Fig. 1. Model with aggregate uncertainty: policy function of unemployed household (state with low TFP). An unemployed household's end-of-period capital holdings, in a state with low TFP, are shown as a function of the household's beginning-of-period capital stock, for different values of the barrier parameter $\phi : \phi = 0, 0.03, 0.2$.

4.1. Policy function for individual capital holdings

We begin by discussing the properties of the (approximate) policy function that expresses the individual's capital stock at the end of period t, k^i_{t+1}, as a function of her beginning-of-period capital stock, k^i_t. We find that the non-negativity constraint $k^i_{t+1} \geq 0$ never binds when the agent is employed ($\varepsilon^i_t = 1$), even when k^i_t is close to zero. However, when the agent is unemployed ($\varepsilon^i_t = 0$), the non-negativity constraint binds when k^i_t is close to zero. The policy function has a slope near unity, in the range where $k^i_{t+1} > 0$.[12]

For the *economy with aggregate uncertainty*, Fig. 1 shows the policy function of an unemployed agent in a state with low TFP (the aggregate capital stock is set at its steady state value). Different values for the barrier parameter ϕ are considered. When $\phi = 0$, the constraint $k^i_{t+1} \geq 0$ is just binding when $k^i_t = 2.09$. For $\phi > 0$, agents are penalized when holding a low capital stock; therefore, the slope of the policy function is lower and, for the unemployed agent, the constraint $k^i_{t+1} \geq 0$ binds at a lower value for k^i_t; when the barrier parameter is at 0.03 (0.2), the non-negativity constraint just binds when $k^i_t = 1.74$ ($k^i_t = 0.94$).

4.2. Euler equation errors

Following den Haan (2009), we define the (absolute) Euler equation error as $|c^i_t - \tilde{c}^i_t|/\tilde{c}^i_t$, where c^i_t is date t consumption implied by an individual agent's optimal policy function (from the solution of the model), while $\tilde{c}^i_t$ is the value of date t consumption implied by the agent's Euler equation (3), given the conditional expectation of her marginal utility of consumption at $t+1$.[13]

We calculate Euler equation errors for an employed agent and for an unemployed agent on a grid of values of the beginning-of-period capital stock k^i_t in the range [0, 100] (step size: 0.01). For the *economy without aggregate uncertainty*, Panel A of Table 1 reports average and maximum Euler equation errors for different value for the barrier parameter ϕ, and also Euler equation errors at selected values of k^i_t. When $\phi = 0$, the average Euler equation error, across all grid points for k^i_t, is 1.64% for an unemployed agent, and 0.03% for an employed agent. Euler equation errors are very small ($10^{-7}-10^{-8}$ for the unemployed agent and $10^{-9}-10^{-11}$ for the employed agent) when the beginning-of-period capital stock is larger than four. Errors are smallest around the steady state capital stock (= 37.99). However, Euler equation errors increase as the beginning-of-period capital stock approaches zero. Errors are largest when the beginning-of-period capital stock is zero (for an employed agent), and at the point where the zero-capital constraint is just binding (when the agent is unemployed).[14]

Recall that when the perturbation solution generates negative individual capital, we enforce the non-negativity constraint by simply setting the capital stock to zero (while consumption is recalculated using the budget constraint); the

[12] In that range, the approximate policy function is linear, as a first-order accurate solution method is used.

[13] We compute conditional expectations using discrete-state Markov probabilities, but (as before) policy functions are based on continuous-state shock processes.

[14] By contrast, errors of deterministic equations (budget constraints) are almost zero across the whole range of individual beginning-of-period capital stocks.

Table 1
Model without aggregate uncertainty: Euler equation errors.

Beginning-of-period capital k_t^i	Unemployed agent			Employed agent		
	$\phi=0$	$\phi=0.03$	$\phi=0.2$	$\phi=0$	$\phi=0.03$	$\phi=0.2$
(A) Benchmark specification with non-negativity constraint for individual capital holdings						
0	0.694	0.657	0.528	0.034	0.021	0.028
0.05	0.651	0.608	0.461	0.033	0.020	0.028
0.1	0.608	0.560	0.394	0.032	0.019	0.028
0.5	0.260	0.170	0.142	0.024	0.010	0.027
1	0.174	0.317	0.422	0.014	0.001	0.026
2	0.985	0.587	0.030	1.1E−09	0.003	0.025
5	1.06E−07	0.0026	0.026	1.0E−09	0.002	0.021
10	1.04E−07	0.0022	0.019	8.6E−10	0.002	0.016
20	9.91E−08	0.0013	0.010	5.2E−10	0.001	0.009
40	9.12E−08	3.7E−06	4.0E−05	7.0E−11	1.2E−04	0.001
60	8.44E−08	0.0010	0.006	5.8E−10	0.001	0.006
80	7.85E−08	0.0018	0.010	1.0E−09	0.002	0.010
100	7.32E−08	0.0025	0.013	1.4E−09	0.002	0.013
Maximum error	1.0157	0.9242	0.5963	0.0339	0.0209	0.0278
k_t^i at maximum error	1.97	1.63	0.84	0	0	0
Average error	0.0164	0.0139	0.0134	0.0003	0.0014	0.0088
(B) Model variant without non-negativity constraint for individual capital holdings						
0	1.084E−07	0.00316	0.0336	1.219E−09	0.0029	0.0278
0.5	1.081E−07	0.00311	0.0327	1.200E−09	0.0028	0.0271
1	1.079E−07	0.00305	0.0318	1.181E−09	0.0028	0.0264
2	1.074E−07	0.00295	0.0301	1.144E−09	0.0027	0.0250
5	1.06E−07	0.00264	0.0256	1.0E−09	0.0024	0.0213

(C) Comparison between linear and quadratic approximations ($\phi=0$)

	Linear	Quadratic	Linear	Quadratic
Maximum error	1.0157	1.0104	0.0339	0.0337
k_t^i at maximum error	1.97	1.96	0	0
Average error	0.0164	0.0164	0.0003	0.0003

Note: The table shows a household's date t Euler equation error (see Section 4.2), for different values of the household's beginning-of-period capital stock k_t^i. Maximum and average errors are calculated on a grid for k_t^i between 0 and 100 (step size: 0.01). Columns (2)–(4) [(5)–(7)] pertain to an unemployed [employed] household. ϕ: barrier parameter.

adjusted capital stock and consumption do not satisfy the Euler equation. This explains the large Euler equation errors for low beginning-of-period capital stocks.

Panel B of Table 1 shows results for a model variant in which the non-negativity constraint on individual capital is removed. In that variant, Euler equation errors are very small ($10^{-7}-10^{-9}$), even for beginning-of-period capital stocks close to zero.

When the barrier parameter is increased, in the model with a non-negativity constraint, then Euler equation errors at low capital stocks fall (as the likelihood of hitting the non-negativity constraint falls), but errors for large capital stocks increase. This is documented in Panel A, where errors for $\phi=0.03$ and 0.2 are reported. In the *economy without aggregate uncertainty*, the barrier parameter $\phi=0.030$ minimizes the sum of Euler equation errors across all grid points for individual capital in the range [0, 100], and across the two values of the individual employment status $\varepsilon_t^i\in\{0,1\}$.[15] The same value of ϕ also minimizes the sum of Euler equation errors in the economy with aggregate uncertainty.[16]

Panel C of Table 1 shows results for a quadratic model solution. That solution generates only slightly smaller Euler equation errors than the linear solution.

4.3. Behavior of individual and aggregate variables

Table 2 shows how some properties of individual and aggregate variables are affected by the value of ϕ, for the *economy with aggregate uncertainty*. All statistics are based on the 10,000 period sequence for the individual employment status and

[15] A simple grid search (with step size 0.001) was used to determine the value of ϕ that minimizes the sum of errors.

[16] In the economy with aggregate uncertainty, we evaluate Euler equation errors on the same grid for individual capital holdings of unemployed and employed agents, and that for states with high and low TFP (we set the aggregate capital stock at its steady state level); $\phi=0.03$ minimizes the sum of errors across all grid points and states.

Table 2
Model with aggregate uncertainty: selected predictions.

Model property	Barrier parameter		
	$\phi = 0$	$\phi = 0.03$	$\phi = 0.2$
Fraction of agents holding zero capital, across all states (%)	1.3426	0.0067	0.00019
Fraction of agents holding zero capital, in state with high TFP (%)	0.8930	0.0032	0.00001
Fraction of agents holding zero capital, in state with low TFP (%)	1.7689	0.0100	0.00035
Std (individual consumption)	0.230	0.151	0.171
Std (individual capital stock)	18.096	7.645	4.190
Std (rental rate)	0.0011	0.0011	0.0010
Std (wage rate)	0.0205	0.0205	0.0167
Std (GDP)	0.1312	0.1315	0.1293
Std (aggregate consumption)	0.0480	0.0481	0.0410

Note: The table shows simulation results based on the 10,000 period sequences for TFP and an individual's employment status, as well as on the initial cross-sectional distribution of capital provided by den Haan et al. (2009). den Haan's (2009) non-stochastic simulation procedure is used to generate next period's capital distribution. Std: standard deviation (of simulated time series).

TFP as well as the initial cross-sectional distribution of capital described in den Haan et al. (2009). We use den Haan's (2009) non-stochastic simulation method based on individual policy functions, to generate the date $t+1$ distribution of capital across agents, given the distribution at t.

When $\phi = 0$, then 1.34% of agents (across all states) are at the zero capital constraint. Raising the barrier parameter to 0.2 decreases the fraction of agents at the zero lower bound to 0.00019%.

The times series volatility of individual capital stocks decreases, when ϕ is increased. As discussed above, with a higher barrier parameter, the slope of the individual policy function (for capital) decreases; thus individual capital becomes less persistent, and its standard deviation falls. The volatility of the rental rate, the wage rate, aggregate GDP and aggregate consumption (implied by the aggregation procedure of the non-stochastic simulation method) is essentially unaffected as ϕ is increased from 0 to 0.03, but it falls slightly when ϕ is increased from 0.03 to 0.2.[17]

5. Results for the setup with a large (but finite) number of agents

This section discusses results for the model variant with a large (but finite) number of agents. For the *economy with aggregate uncertainty*, Fig. 2 (Panel A) shows an unemployed household's decision rule for capital holdings at the end of period t, k^i_{t+1} (as a function of k^i_t), in a state with low TFP ($a_t = 0.99$).[18] The barrier parameter is set at $\phi = 0.03$. As the number of agents rises from $N = 10$ to $N = 70$, the policy function approaches the policy function of the "representative-agent setup" considered in the previous section.[19] Policy functions in other individual/aggregate states (employed agent, high TFP) have the same property.

Panel B of Fig. 2 shows that the time-series volatility of the wage rate and of the rental rate of capital falls as the number of agents rises (again, $\phi = 0.03$ is used).[20] The time-series volatility of *individual* consumptions and capital holdings stabilizes once N is increased above 40 (not reported in figure).

6. Conclusion

This paper has shown how to apply a perturbation method to solve an incomplete-markets model with a continuum of heterogenous agents and occasionally binding inequality constraints. Traditionally, perturbation methods have not been used to solve this type of model. To make the problem here amenable to a perturbation method, we replace the heterogenous-agent model by an economy with a single agent who faces factor prices generated by a simple representative-agent setup; we also consider a model variant with a large (but finite) number of agents. We use a "barrier method" to obtain a problem with only equality constraints. The accuracy of the solution can be low when the individual capital stock is very close to zero; however, accuracy is satisfactory for larger individual capital stocks. The key benefit of the method presented here is that it is much easier to use, and much faster,

[17] In contrast, ϕ does not affect the volatility of aggregate quantities generated by the "representative-agent setup."

[18] The beginning-of-period capital holdings and employment of the *remaining* agents are set at steady state levels.

[19] In a heterogenous agent setting, den Haan (1996) investigated how model statistics change as the number of agents increases. den Haan (1997) compared a model with a continuum of agents to a model with a large number of agents ($N = 100,000$) and found that both models have similar properties.

[20] The standard deviations reported for a given value of N are averages of standard deviations computed for each of 100 simulation runs with 1,000 periods (first 100 periods removed). Each simulation is based on individual employment and TFP series generated using the discrete-state Markov processes.

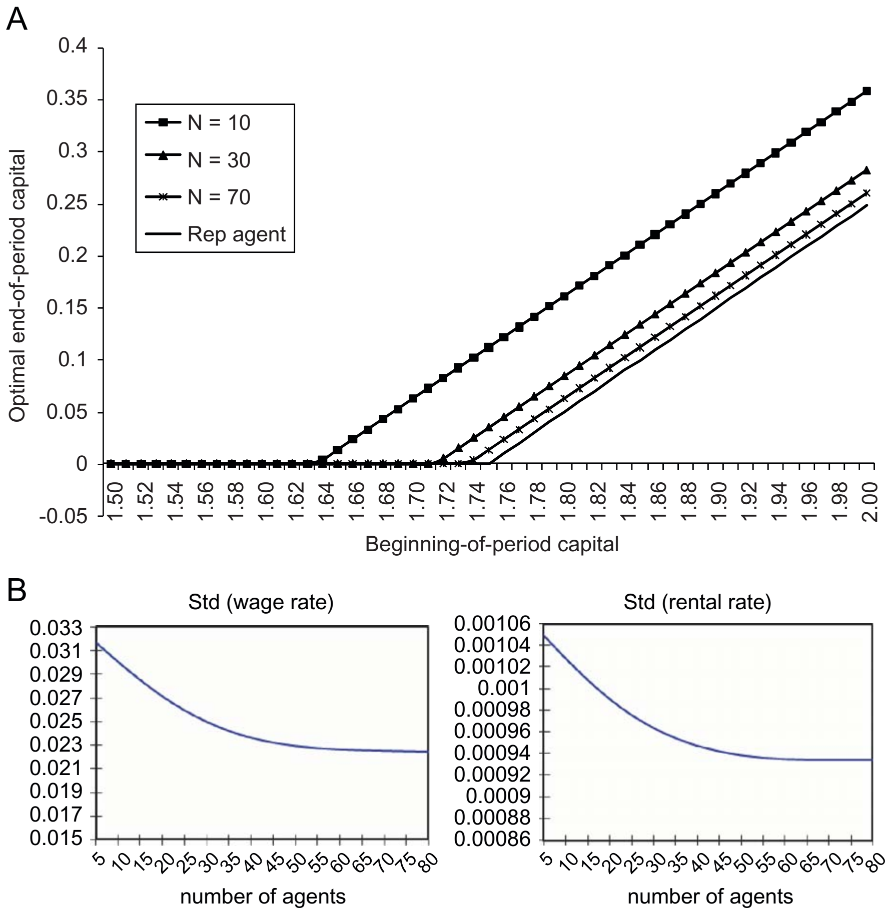

Fig. 2. *N*-agent model with aggregate uncertainty: quantitative predictions. Panel A shows an unemployed household's end-of-period capital holdings, in a state with low TFP, as a function of the household's beginning-of-period capital stock, in the *N*-agent model (with $N = 10$, 30, 70), and in the "representative-agent" model (see line labeled "Rep agent"). The barrier parameter is set at $\phi = 0.03$. Panel B shows the standard deviations (Std) of the wage rate and of the rental rate of capital, as a function of *N*.

than other techniques. Hence, the results here suggest that perturbation methods deserve to be considered by researchers who study heterogeneous-agent economies, as well as economies with occasionally binding inequality constraints.

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