Rational Bubbles in Non-Linear Business Cycle Models: Closed and Open Economies

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21 July, 2020

● Contribution: construct BOUNDED "RATIONAL BUBBLES" in standard NON-LINEAR RBC macro models

● Rational bubbles (Blanchard, 1979): multiple equilibria arising from absence of transversality condition (TVC) for asset

►Blanchard's rational bubble pertains to model in which TVC pins down unique stable saddle path.

► Rational bubble (no TVC) imply that economy deviates from saddle path, but may stochastically revert to saddle path: Booms followed by sudden busts

● **Lack of TVC can be due to OLG structure with finitely-lived households**

● **Key contribution: without TVC, the non-linear models here have multiple bounded solutions**

BOUNDEDNESS reflects NON-LINEAR EFFECTS

● **Bounded rational bubbles feature recurrent boom-bust cycles in investment & output: persistent expansions followed by abrupt contractions**

Rational bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment

● **Bounded rational bubbles in non-linear models: novel driver of business cycles**

Non-linear DSGE models driven just by stationary bubbles can generate persistent fluctuations of real activity & capture key business cycle stylized facts

● **BOUNDEDNESS of rational bubbles reflects NON-LINEAR EFFECTS:**
Rational bubbles in LINEARIZED models are explosive: expected trajectories tend to
$\pm\infty$

● **But: rational bubbles in NON-LINEAR models can be bounded (stable)**

$\Rightarrow$ Blanchard & Kahn (1980): conditions for existence of unique **stable** solution of **linear(ized)** models are IRRELEVANT for non-linear models

Non-linear DSGE models have more stationary solutions than you think!

Blanchard (1979): rational bubbles in linear asset pricing models without TVC

$E_t y_{t+1} = \lambda \cdot y_t, \;\; \lambda > 1$**;** $\quad y_t$: scalar jump variable

Unique stable solution: $y_t = 0$

Blanchard (1979)

Bubble: $y_{t+1} = (\lambda/(1-\pi)) \cdot y_t$ with probability $1-\pi$

$y_{t+1} = 0$ with probability π

$\lim_{s\to\infty} E_t y_{t+s} = \pm\infty$ if $y_t \neq 0$

expected path of bubble diverges to ±∞

Blanchard (1979): powerful narrative, much cited (mainly in finance), but NO influence on structural macro modeling.

This paper: shows how to embed Blanchard (1979) bubble into DSGE model

Key contribution: in non-linear model BUBBLES CAN BE BOUNDED

Expected path of bubbles in non-linear DSGE studied here do NOT diverge to $\pm\infty$

WHY BOUNDEDNESS OF RATIONAL BUBBLES IS IMPORTANT:

● Explosive bubbles trajectories are problematic:

► accuracy of linear(ized) models breaks down far from point of approximation; non-negativity & technological feasibility constraints may be violated

Example: with decreasing returns to capital, explosive trajectory of capital & output is INFEASIBLE

⇒ LINEARIZED MODELS UNSUITABLE FOR ANALYZING RATIONAL BUBBLES

- By contrast: non-linear analysis here takes non-negativity constraints, decreasing returns & risk aversion into account
- Decreasing returns & risk aversion generate stabilizing forces that prevent explosive trajectories
- Stationary rational bubbles in non-linear models are one-sided (capital over-accumulation, but no under-accumulation) [By contrast: Blanchard bubbles in linear models can be positive or negative]

● Rational bubbles in <u>non-linear</u> model can induce fluctuations that are <u>close to deterministic steady state most of the time</u>

⇒ unconditional mean of endogenous variables close to deterministic steady state

Note: Can construct linearized DSGE models with non-explosive sunspot equilibria:

$E_t y_{t+1} = \lambda \cdot y_t$ **need** $|\lambda| \leq 1$. $\Rightarrow y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}$ is stationary solution for <u>any</u> $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1} = 0$

Needed ingredients:

- Increasing returns, externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999))
- Financial frictions (e.g., Martin and Ventura (2018))
- Overlapping generations (e.g., Woodford (1986), Galí (2018))

Specific assumptions & calibrations that deliver $|\lambda|<1$ can be debatable & fragile (e.g. in standard OLG model: need r≤g)

<u>By contrast, paper here argues that very standard DSGE models with $|\lambda|>1$ can deliver stationary sunspot equilibria, if non-linearities are considered.</u>

Example I:
Long-Plosser RBC model with ration. bubble

$u(C)=\ln(C)$; $C_t+K_{t+1}=Y_t$; $Y_t=F(K_t)\equiv(K_t)^{\alpha}$, $0<\alpha<1$

Euler equation: $\beta E_t\{u'(C_{t+1})/u'(C_t)\}\cdot F'(K_{t+1})=1$

$\Rightarrow \beta E_t\{C_t/C_{t+1}\}\cdot \alpha Y_{t+1}/K_{t+1}=1$

$\Rightarrow \beta E_t\{(Y_t-K_{t+1})/(Y_{t+1}-K_{t+2})\}\cdot \alpha Y_{t+1}/K_{t+1}=1$

$\Rightarrow \alpha\beta\cdot E_t\{(1-K_{t+1}/Y_t)/(1-K_{t+2}/Y_{t+1})\}\cdot Y_t/K_{t+1}=1$

$\Rightarrow \alpha\beta\cdot E_t\{(1-Z_t)/(1-Z_{t+1})\}/Z_t=1$,

$Z_t\equiv K_{t+1}/Y_t$: investment/output ratio

Textbook solution: $Z_t=\alpha\beta$

$$\alpha\beta \cdot E_t\{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1$$

Linearization around $Z=\alpha\beta$:

$$E_t z_{t+1} = \lambda \cdot z_t, \quad z_t \equiv Z_t - Z; \quad \lambda \equiv 1/(\alpha\beta) > 1.$$

$\Rightarrow z_t = 0$ is <u>unique</u> non-explosive solution of <u>linearized</u> model.

But: **<u>non-linear</u> model has other stationary solutions.**

$$\alpha\beta \cdot \{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1 + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

$$\Rightarrow Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1 + \varepsilon_{t+1}).$$

Z_{t+1} increasing & strictly concave in Z_t & ε_{t+1}

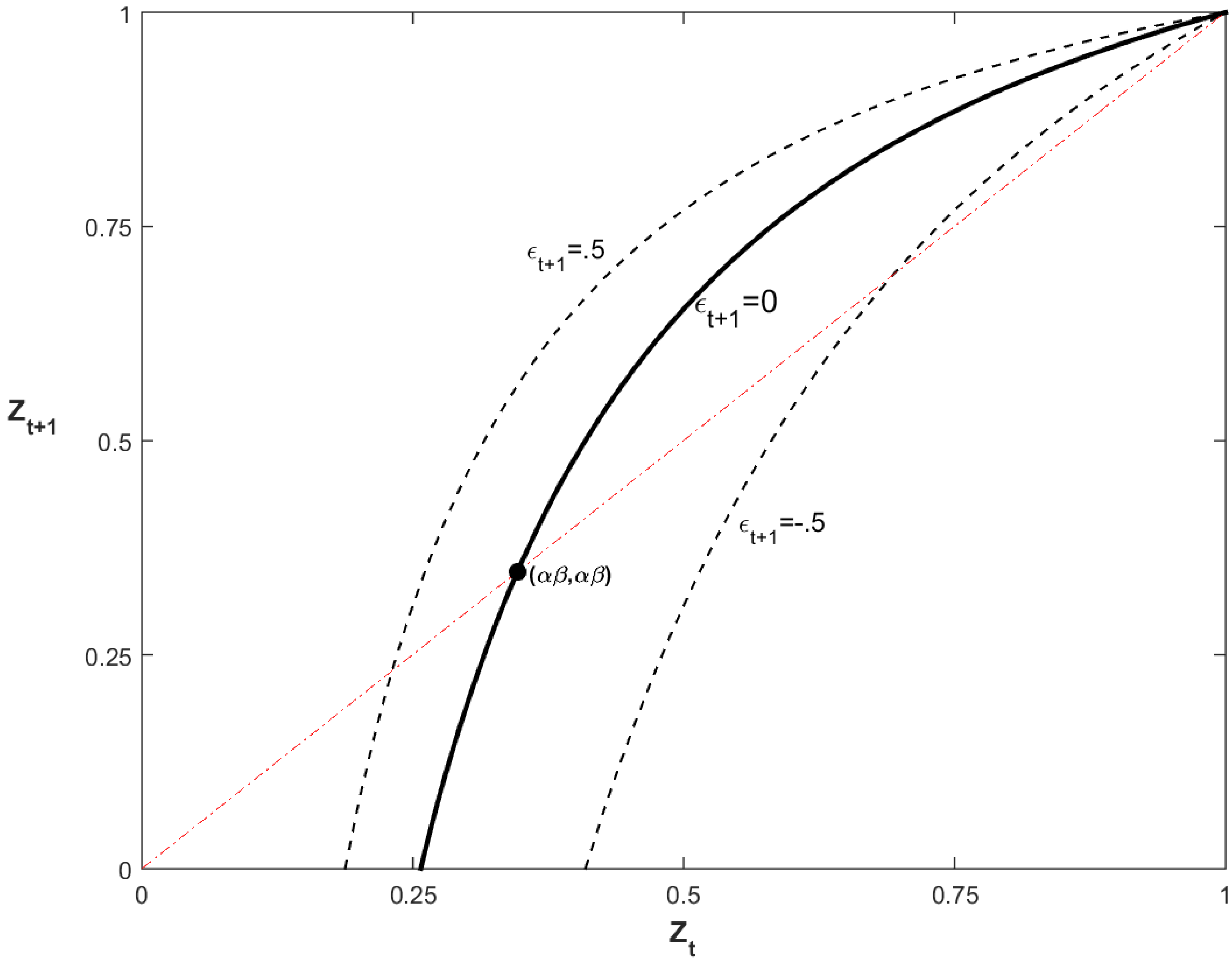


Fig.1. Long & Plosser model: investment/output ratio at t+1, Z_{t+1}, **as function of** Z_t **for** $\varepsilon_{t+1} \in \{-0.5; 0; 0.5\}$

$$Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1+\varepsilon_{t+1}); \quad \alpha = 1/3, \beta = 0.99.$$

• Intuition why Z_{t+1} is increasing in Z_t:

$$C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t), \quad F' > 0, \quad F'' < 0$$

$\beta\{[E_t u'(C_{t+1})]/u'(C_t)\}\cdot F'(K_{t+1}) = 1$; $u''' > 0$ (CRRA)

Sunspot: assume $K_{t+1}\uparrow \Rightarrow C_t\downarrow \; u'(C_t)\uparrow$,

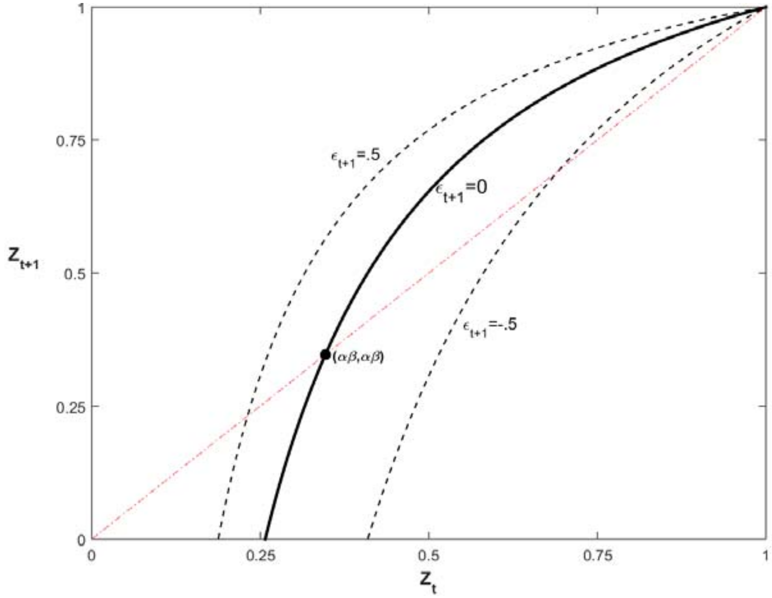
$F'(K_{t+1})\downarrow$ Euler eqn requires:

$$E_t u'(C_{t+1}) = E_t u'(F(K_{t+1}) - K_{t+2})\uparrow$$

• In <u>deterministic</u> economy: need $C_{t+1}\downarrow \& K_{t+2}\uparrow$

K_{t+2} has to rise <u>**more**</u> than K_{t+1} ! $\Rightarrow$ <u>**K diverges**</u>

Thus: $Z_t \uparrow \;\Rightarrow\; Z_{t+1}\uparrow$



$$Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1}) \equiv 1-\alpha\beta(1/Z_t-1)/(1+\varepsilon_{t+1})$$

- When $Z_t<\alpha\beta$, the model can hit zero-capital corner solution in later periods $\Rightarrow$ restrict attention to solutions with $Z_\tau \in [\alpha\beta,1) \;\; \forall\, \tau$

- Support of ε_{t+1} has to be bounded below: $\varepsilon_{t+1} \geq -1+[\alpha\beta/(1-\alpha\beta)]\cdot[1/Z_t-1]$
$\Rightarrow$ distribution of ε_{t+1} must depend on Z_t !

- Let ε_{t+1} only takes two values: $-\overline{\varepsilon}_t$ and $\overline{\varepsilon}_t \cdot \pi_t/(1-\pi_t)$ with probabilities π_t and $1-\pi_t$, respectively, $\overline{\varepsilon}_t \in [0,1)$ $\Rightarrow$ Z_{t+1} takes two values:

$$Z^L_{t+1}\equiv\Lambda(Z_t,-\overline{\varepsilon}_t) \;\&\; Z^H_{t+1}\equiv\Lambda(Z_t,\overline{\varepsilon}_t\pi_t/(1-\pi_t)) \text{ with } Z^L_{t+1}\leq Z^H_{t+1}\leq 1.$$

Specification à la Blanchard (1979):

Given Z_t^L, next period two values possible

- $Z_{t+1}^L = \alpha\beta + \Delta,\ \Delta = 10^{-6}$ **with probability** $\pi = 0.5$

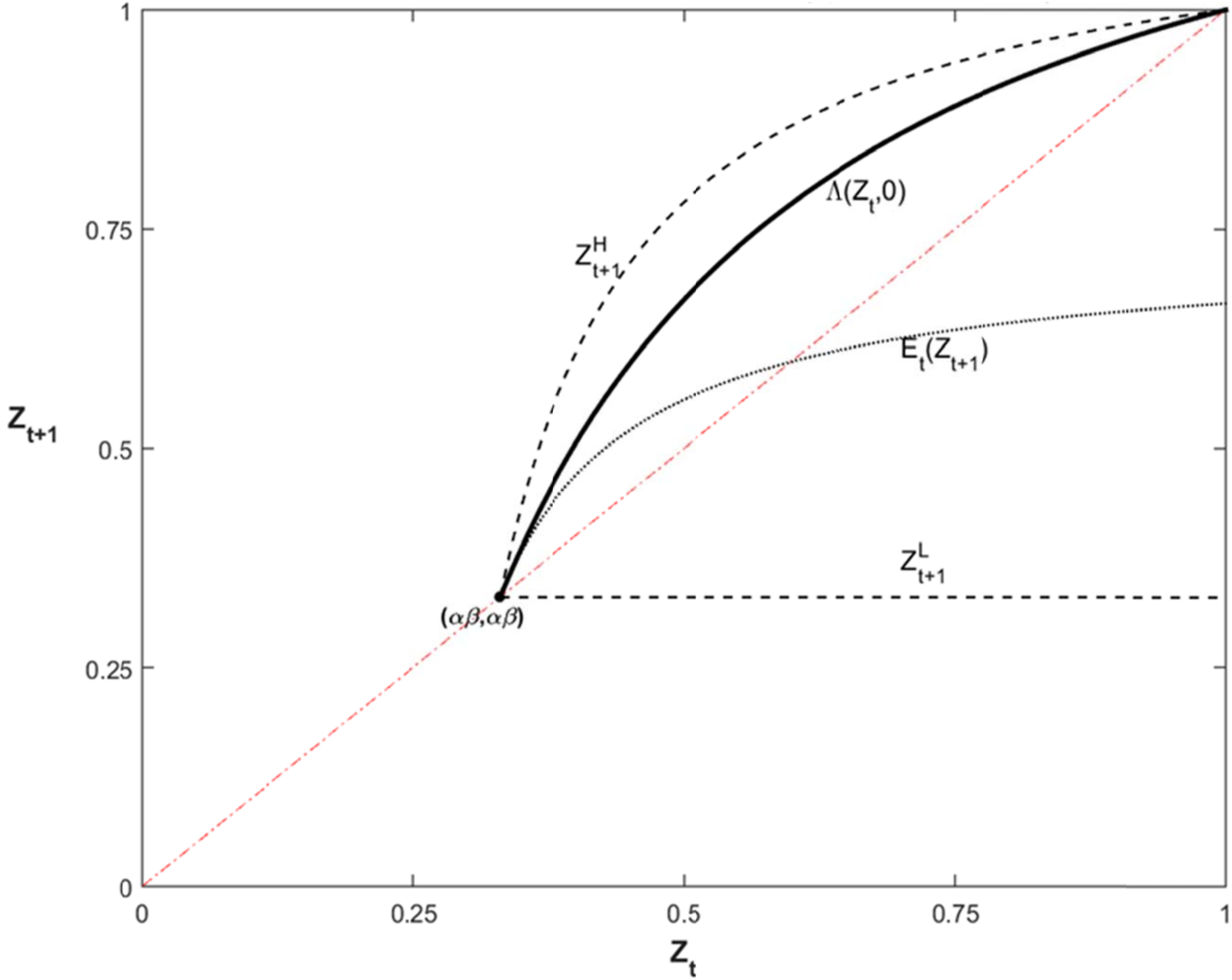
(When $\Delta = 0$, then $Z = \alpha\beta = 0.346$ is absorbing state; thus set $\Delta > 0$)

- $Z_{t+1}^H > \alpha\beta + \Delta$ **with prob.** $1 - \pi$

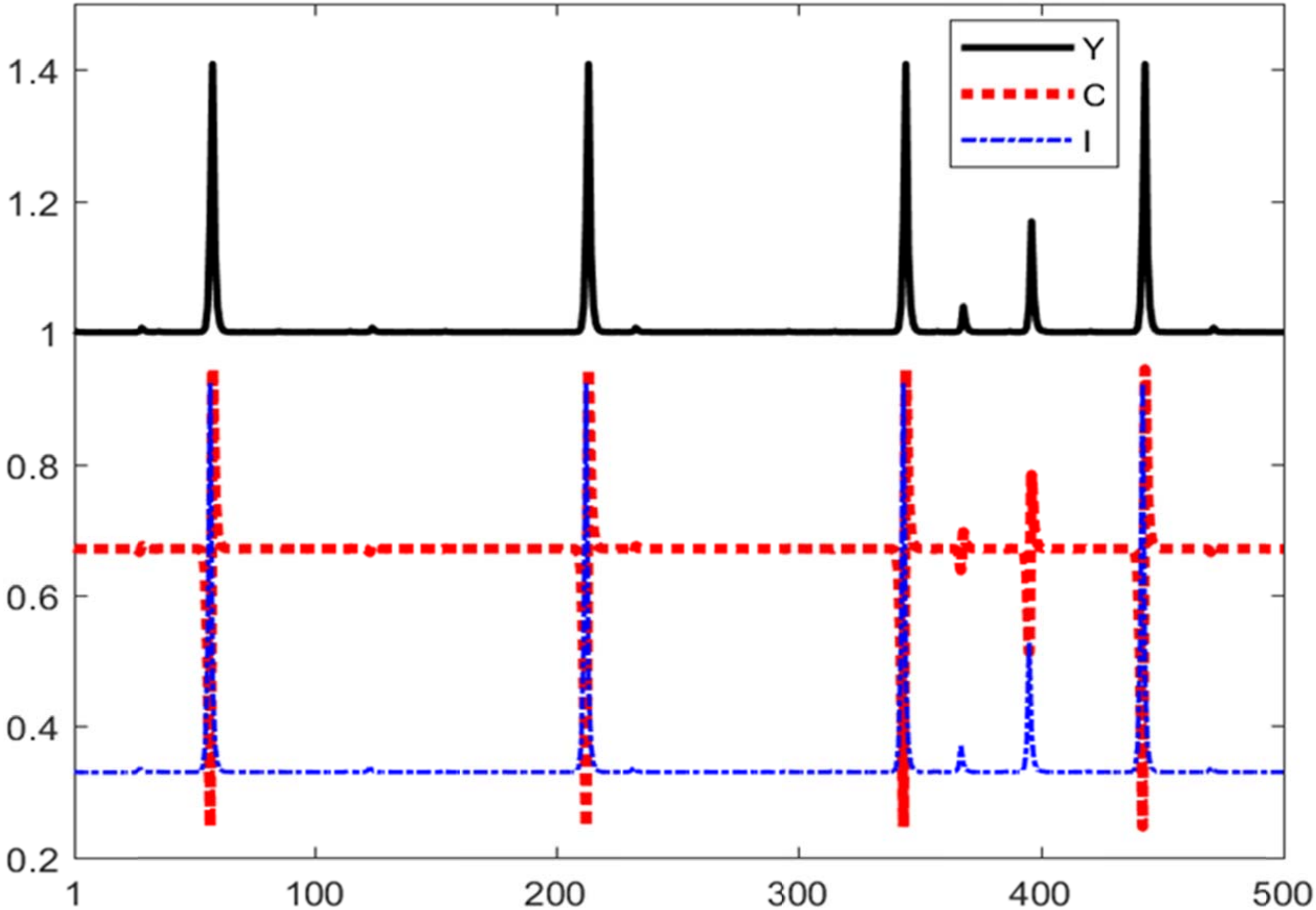
Z_{t+1}^H **pinned down by Euler eqn:**

$$\pi \cdot \alpha\beta \cdot \{(1 - Z_t)/(1 - Z_{t+1}^L)\}/Z_t +$$

$$(1 - \pi) \cdot \alpha\beta \cdot \{(1 - Z_t)/(1 - Z_{t+1}^H)\}/Z_t = 1$$



1
0.75
0.5
0.25
0
0
0.25
0.5
0.75
1
Z_{t+1}
Z_t
Z_{t+1}^H
$\Lambda(Z_t,0)$
$E_t(Z_{t+1})$
Z_{t+1}^L
$(\alpha\beta,\alpha\beta)$



Simulated series with const. probability: $\pi=0.5$

Simulated output (Y), consumption (C) and investment (I) normalized by steady state output

Table 1. Long-Plosser model (closed economy) with bubbles: business cycle statistics

	Standard dev. %			Corr. with Y		Autocorrelations			Mean [% deviation from SS]			
	Y	*C*	*I*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*	*Z*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
(a) Predicted business cycle statistics												
	1.14	2.35	3.41	0.38	0.43	0.42	-0.19	0.42	0.54	-0.34	2.35	1.01
(b) Historical business cycle statistics												
	1.47	1.19	4.96	0.87	0.92	0.87	0.89	0.92				

Notes: Row (a) reports simulated business cycle statistics for a Long-Plosser economy with bubbles (no transversality condition); see Sect. 2 of paper. *Y*: output; *C*: consumption; *I*: investment; *Z*: investment/output ratio.

Novel result about OLG economy:
Assume: (I) Complete financial market allow efficient risk sharing across all generations alive at dates *t* **and** *t*+1
(II) Each generation receives wealth endowment such that consumption by newborns is time-invariant share of aggregate consumption. (Under log-utility: wealth endowment of newborns has to be time-invariant share of total wealth)
THEN
an ‘aggregate’ Euler equation holds that is identical to the Euler equation of a representative infinitely lived household:

$$\beta E_t\{u'(C_{t+1})/u'(C_t)\}MPK_{t+1}=1$$

BUT: there is no TVC in the OLG economy!

OLG structure with efficient intergenerational risk sharing:

justification for macro models that lacks a TVC, but whose other equilibrium conditions are identical to those of standard business cycle models (that assume infinitely lived agents)

Example II: RBC model with incomplete capital depreciation & endogenous labor

$U(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t),\quad \Psi>0$, L_t: hours worked

$$C_t+K_{t+1}=Y_t+(1-\delta)K_t\ ,\quad Y_t=\theta(K_t)^{\alpha}(L_t)^{1-\alpha}$$

• FOCs: $C_t\Psi/(1-L_t)=(1-\alpha)\theta(K_t)^{\alpha}(L_t)^{-\alpha}$

$$E_t\beta\{C_t/C_{t+1}\}(\alpha\theta(K_{t+1})^{\alpha-1}(L_{t+1})^{1-\alpha}+1-\delta)=1$$

• Using static efficiency conditions can express *C* & *L* as functions of capital:

$$C_t=\gamma(K_{t+1},K_t)\ ,\ L_t=\eta(K_{t+1},K_t)$$

Can writer Euler equation as:

$$E_t[\beta\{\gamma(K_{t+1},K_t)/\gamma(K_{t+2},K_{t+1})\}(\alpha\theta(K_{t+1})^{\alpha-1}(\eta(K_{t+2},K_{t+1}))^{1-\alpha}+1-\delta)]=1$$

Euler equation:

$$E_t H(K_{t+2}, K_{t+1}, K_t) = 1$$

No bubble solution (TVC): described by policy function $K_{t+1} = \lambda(K_t)$

so that $E_t H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1$

Consider **bubble equilibria** such that, for any *t*, K_{t+1} takes one of two values $K_{t+1} \in \{K_{t+1}^{L}, K_{t+1}^{H}\}$ with exogenous probabilities π and $1-\pi$, where $K_{t+1}^{L} = \lambda(K_t)e^{\Delta}$;
$\Delta > 0$: small positive constant

'L' is 'bust' state, in which capital stock set at t reverts to value close to 'no-bubble' decision rule

Euler equation
$E_t H(K_{t+2}, K_{t+1}, K_t) = 1$
becomes:
$\pi H(\lambda(K_{t+1})e^{\Delta}, K_{t+1}, K_t) + (1-\pi) \cdot H(K_{t+2}^{H}, K_{t+1}, K_t) = 1$

Economy evolves as follows:

At date t: random draw (with probab. π, 1-π) determines $K_{t+1} \in \{K^L_{t+1}, K^H_{t+1}\}$ where $K^L_{t+1} = \lambda(K_t)e^{\Delta}$

Euler equation between t and t+1 determines K^H_{t+2}:

$$\pi H(\lambda(K_{t+1})e^{\Delta}, K_{t+1}, K_t) + (1-\pi)\cdot H(K^H_{t+2}, K_{t+1}, K_t) = 1$$

Etc. in all subsequent periods.

See paper for: • Existence proof of sunspot equilibrium: need $\Delta>0$. Then $K^L_{t+1} < K^H_{t+1}$

● Analysis with stochastic TFP

Numerical simulations

β=0.99; α=1/3; δ=0.025;

Labor supply elasticity (at steady state) = 1.

- Log utility (unit risk aversion, RA): $\ln(C_t)$
- 'High Risk Aversion' utility: $\ln(C_t - \overline{C})$, $\overline{C} > 0$

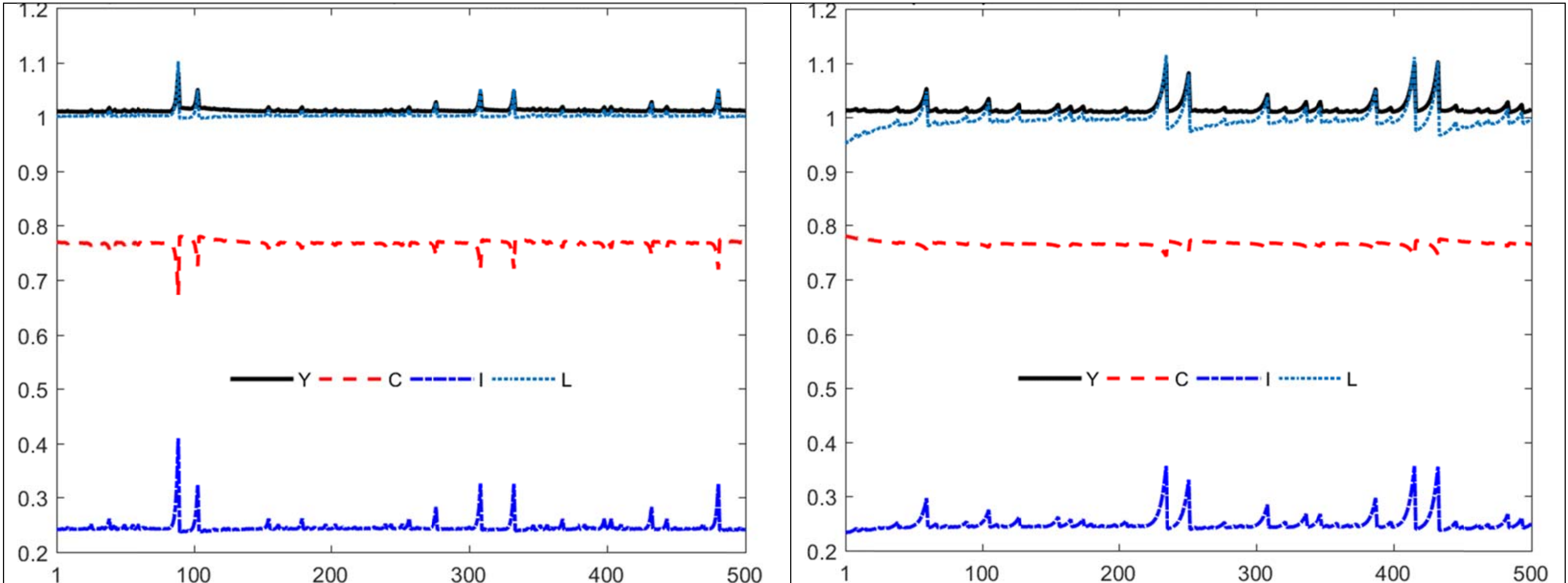
Parameters of bubble process:

Δ=0.001

Bust probabilities: π=0.5, π=0.2.

RBC model (incomplete capital deprec.) with bubbles: predicted business cycle statistics

	Unit Risk aversion		High RA		
	π=0.5	π=0.2	π=0.5	π=0.2	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
Standard deviations [in %]					
Y	0.49	1.16	0.68	1.43	1.81
C	1.08	2.63	0.29	0.61	1.35
I	4.29	9.38	3.22	6.51	5.30
L	0.74	1.73	1.04	2.18	1.79
Correlations with GDP					
C	-0.97	-0.95	-0.99	-0.98	0.88
I	0.98	0.96	0.99	0.99	0.80
L	0.99	0.97	0.99	0.99	0.88
Autocorrelations					
Y	0.36	0.63	0.35	0.62	0.84
C	0.33	0.60	0.35	0.62	0.80
I	0.36	0.63	0.37	0.64	0.87
L	0.34	0.61	0.35	0.62	0.88
Means [% deviation from steady state]					
Y	1.41	2.80	1.25	2.12	--
C	0.73	1.39	0.33	0.55	--
I	3.62	7.33	4.22	7.19	--
L	0.36	0.74	-0.02	-0.02	--
Mean (capital income – investment)/GDP [in %]					
	9.12	8.75	8.93	8.54	13.42
Fraction of periods with (capital income > investment) [in %]					
	99.2	96.3	99.5	97.7	100



Unit Risk Aversion; Bust probability: π=0.5

High Risk version; π=0.2

Non-linear RBC model (incompl. capital depreciation) driven by bubbles

Simulated GDP, C and I series normalized by steady state GDP. Hours worked (L) normalized by steady state hours.

CONCLUSIONS

- **Bounded rational bubbles exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions.**
- **Rational bubbles: novel source of business cycles**
- **Induce boom-bust cycles: persistent investment and output expansions followed by abrupt contractions**