

# **Stationary Rational Bubbles in Non-Linear Business Cycle Models**

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April 24, 2019

**Main result:**

**non-linear DSGE models have more stationary equilibria than you think!**

Blanchard & Kahn (1980): conditions for existence of unique **stationary** solution of **linear(ized)** models are IRRELEVANT for non-linear models

**This paper shows: stationary sunspot equilibria exist in standard non-linear DSGE models, even when the linearized versions of these models have unique solution**

**⇒ DSGE models may have multiple equilibria, if non-linearity taken into account**

**These sunspot equilibria look like ‘bubbles’: Economy temporarily diverges from no-sunspots trajectory, before reverting abruptly towards no-sunspots trajectory**

**“Divergent behavior”: similar to ‘rational’ bubbles in linear models (Blanchard (1979))**

**Big difference compared to Blanchard bubbles: bubbles here are STATIONARY.**

$$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1$$

Unique stable solution:  $y_t = 0$

**Blanchard (1979) :**

Bubble:  $y_{t+1} = (\lambda / (1 - \pi)) \cdot y_t$  with probability  $1 - \pi$

$y_{t+1} = 0$  with probability  $\pi$

$$\lim_{s \rightarrow \infty} E_t y_{t+s} = \pm \infty$$

**expected path of bubble diverges to  $\pm \infty$**

**Expected path of bubbles in non-linear DSGE described here do NOT diverge to  $\pm \infty$**

# Note: Can construct DSGE models whose linearized versions have sunspots:

$E_t y_{t+1} = \lambda \cdot y_t$  **need**  $|\lambda| < 1$ .  $\Rightarrow y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}$  is stationary solution for any  $\{\varepsilon_{t+1}\}$  with  $E_t \varepsilon_{t+1} = 0$

Needed ingredients:

- Increasing returns, externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999))
- Financial frictions (e.g., Martin and Ventura (2018))
- Overlapping generations (e.g., Galí (2018))

Specific assumptions & calibrations that deliver  $|\lambda| < 1$  can be debatable & fragile (e.g. in standard OLG model: need dynamic inefficiency,  $r \leq g$ )

**By contrast, paper here argues that very standard DSGE model can deliver stationary sunspot equilibria, if non-linearities are considered.**

## Basic intuition I:

Consider model with just 1 non-predetermined variable (no exogenous driver)

$$E_t G(Y_{t+1}, Y_t) = 0$$

Linearization (around steady state) gives:

$$E_t y_{t+1} = \lambda \cdot y_t, \quad y_t \equiv Y_t - Y^{SS}$$

Linearized model has unique non-explosive solution iff  $|\lambda| < 1$ . That unique solution is:  $y_t = 0$   
(Blanchard & Kahn (1980), Prop. 1)

I show: even when  $|\lambda| > 1$ , the **non-linear** model can have stationary sunspot equilibrium

$$E_t G(Y_{t+1}, Y_t) = 0 \quad \Rightarrow \quad G(Y_{t+1}, Y_t) = \varepsilon_{t+1} \quad \text{for} \quad E_t \varepsilon_{t+1} = 0$$

$$\Rightarrow Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1}) . \quad \varepsilon_{t+1} : \text{“sunspot shock”}$$

Paper shows: even if  $|\Lambda_Y| > 1$ , there may exist a process  $\{\varepsilon_{t+1}\}$  with  $E_t \varepsilon_{t+1} = 0$  such that  $\{Y_{t+1}\}$  is stationary.

Note: when white noise  $\{\varepsilon_{t+1}\}$  is fed into  $Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1})$ , then  $\{Y_{t+1}\}$  diverges if  $|\Lambda_Y| > 1$ .

Key requirements for stationary solution:

- $Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1})$  has to be **NON-LINEAR** in  $\varepsilon_{t+1}$
- **Distribution of  $\varepsilon_{t+1}$  has to depend on  $Y_t$**

$$Y_{t+1} \cong \Lambda(Y_t, 0) + \Lambda_{\varepsilon}(Y_t, 0) \cdot \varepsilon_{t+1} + \frac{1}{2} \Lambda_{\varepsilon\varepsilon}(Y_t, 0) \cdot (\varepsilon_{t+1})^2$$

$$E_t Y_{t+1} \cong \Lambda(Y_t, 0) + \frac{1}{2} \Lambda_{\varepsilon\varepsilon}(Y_t, 0) \cdot E_t (\varepsilon_{t+1})^2$$

Let  $E_t (\varepsilon_{t+1})^2 = f(Y_t) \geq 0$ . If  $\Lambda_{\varepsilon\varepsilon}(Y_t, 0) \neq 0$  then can set  $E_t (\varepsilon_{t+1})^2 = f(Y_t)$  such that  $|dE_t Y_{t+1} / dY_t| < 1$ :

“MEAN REVERSION”

Example:  $\Lambda_Y(Y_t, 0) > 1$ ,  $\Lambda_{\varepsilon\varepsilon}(Y_t, 0) < 0$ . Then need

$f'(Y_t) > 0$  for mean reversion:  $E_t (\varepsilon_{t+1})^2$  must be increasing in  $Y_t$ .



## Basic intuition II: RBC model

$$C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t), \quad F' > 0, \quad F'' < 0$$

$$\beta \{ [E_t u'(C_{t+1})] / u'(C_t) \} \cdot F'(K_{t+1}) = 1; \quad \text{assume } u''' > 0 \text{ (CRRA)}$$

Sunspot: assume  $K_{t+1} \uparrow \Rightarrow C_t \downarrow, u'(C_t) \uparrow, F'(K_{t+1}) \downarrow$

Euler eqn requires:  $E_t u'(C_{t+1}) = E_t u'(F(K_{t+1}) - K_{t+2})$

• In deterministic economy: need  $C_{t+1} \downarrow$  &  $K_{t+2} \uparrow$   
 $K_{t+2}$  has to rise more than  $K_{t+1}$  !  $\Rightarrow$  **K diverges**

• With stochastic sunspot:  $K_{t+2}$  random.

$u'(C_{t+1})$  is convex in  $K_{t+2} \Rightarrow$  if  $Var_t(K_{t+2})$  rises,

$E_t K_{t+2}$  may rise less than  $K_{t+1}$ :

$\Rightarrow$  **possibility of mean reversion**

Several of the models considered below are usually presented as outcome of decision problem of an infinitely-lived representative agent.

**Bubbles violate the transversality condition (TVC) of infinitely lived household:**

$$\lim_{\tau \rightarrow \infty} \beta^\tau E_t u'(C_{t+\tau}) K_{t+\tau+1} > 0$$

This paper disregards TVC:

1) Lansing (2010) disregards the TVC in a Lucas-style asset pricing models with bubbles, arguing that “agents are forward-looking but not to the extreme degree implied by the transversality condition”

2) In richer models with heterogeneous agents and distortions: equilibrium is not solution of decision problem of representative agent.

Detection of TVC violations in stochastic economies: virtually impossible, even with very long simulation runs (billions of periods):

States with very low consumption might only occur with extremely small probabilities.

3) Assume OLG population structure with agents who live  $N < \infty$  periods: then the TVC (infinite horizon) does not hold

Novel result about OLG economy:

- (i) if there is a complete financial market that allows all generations alive at both dates  $t$  and  $t+1$
- (ii) if the generation born at date  $t$  receives a wealth endowment that is a constant share  $1/N$  of aggregate date  $t$  wealth (across all generations)

THEN

an 'aggregate' Euler equation holds that is identical to the Euler equation of a representative infinitely lived household:

$$\beta E_t \{u'(C_{t+1}) / u'(C_t)\} MPK_{t+1} = 1$$

BUT: there is no TVC in the OLG economy!

**OLG structure with efficient  
intergenerational risk sharing:**

**justification for considering macro models  
that lack a TVC, but whose other equilibrium  
conditions are identical to those of standard  
business cycle models (that assume  
infinitely lived agents)**

# Detailed Example I:

## Long-Plosser RBC model with sunspots

$$u(C)=\ln(C); C_t+K_{t+1}=Y_t; Y_t=F(K_t)\equiv(K_t)^\alpha, 0<\alpha<1$$

$$\text{Euler equation: } \beta E_t \{u'(C_{t+1})/u'(C_t)\} \cdot F'(K_{t+1}) = 1$$

$$\Rightarrow \beta E_t \{C_t / C_{t+1}\} \cdot \alpha Y_{t+1} / K_{t+1} = 1$$

$$\Rightarrow \beta E_t \{(Y_t - K_{t+1}) / (Y_{t+1} - K_{t+2})\} \cdot \alpha Y_{t+1} / K_{t+1} = 1$$

$$\Rightarrow \alpha\beta \cdot E_t \{(1 - K_{t+1}/Y_t) / (1 - K_{t+2}/Y_{t+1})\} \cdot Y_t / K_{t+1} = 1$$

$$\Rightarrow \alpha\beta \cdot E_t \{(1 - Z_t) / (1 - Z_{t+1})\} / Z_t = 1,$$

$$Z_t \equiv K_{t+1} / Y_t : \text{ investment/output ratio}$$

$$\text{Textbook solution: } Z_t = \alpha\beta$$

$$\alpha\beta \cdot E_t \left\{ \frac{(1-Z_t)}{(1-Z_{t+1})} \right\} / Z_t = 1$$

Linearization around  $Z = \alpha\beta$ :

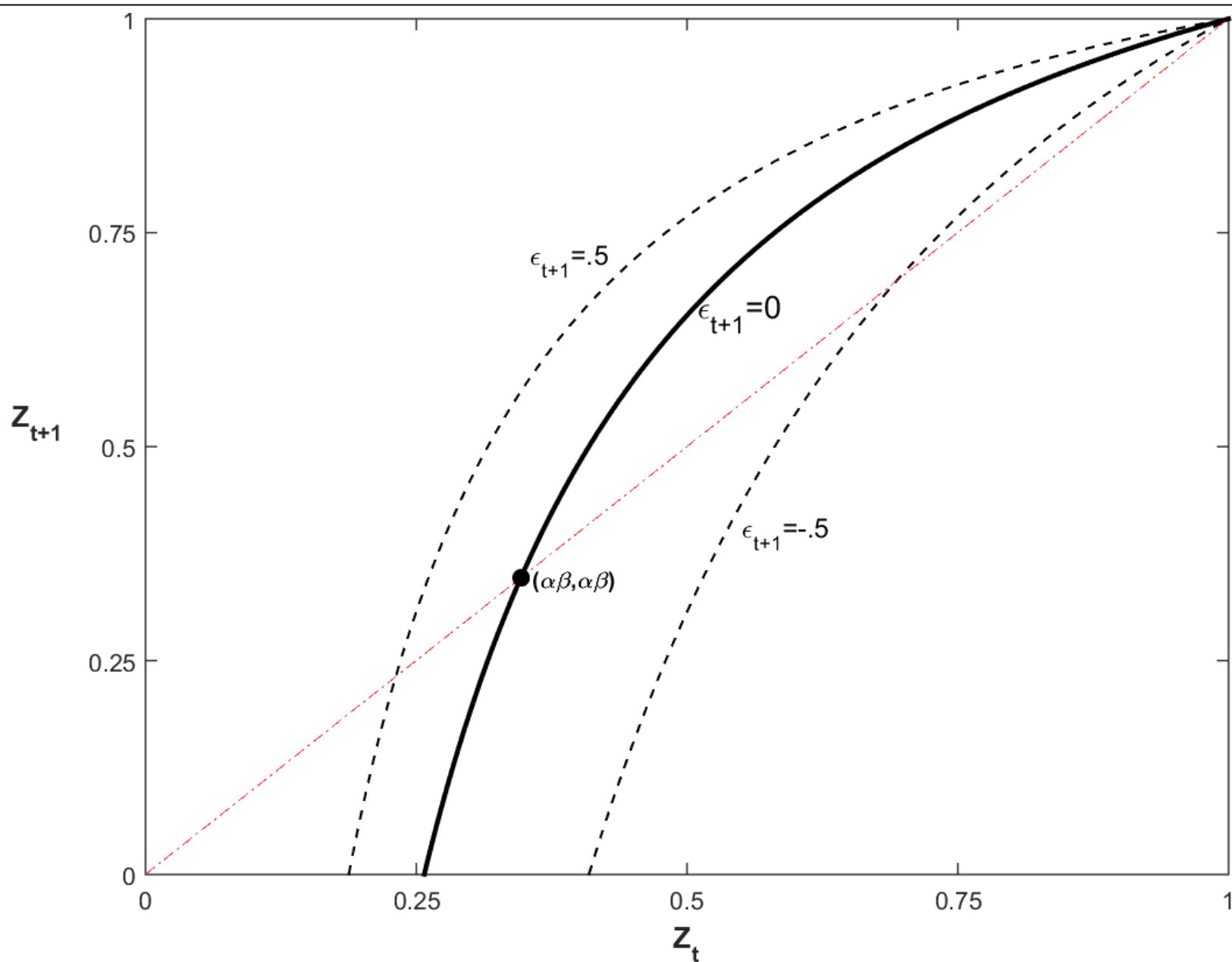
$$E_t z_{t+1} = \lambda \cdot z_t, \quad z_t \equiv Z_t - Z; \quad \lambda \equiv 1/(\alpha\beta) > 1.$$

$\Rightarrow z_t = 0$  is unique non-explosive solution of linearized model.

But: non-linear model has other stationary solutions.

$$\alpha\beta \cdot \left\{ \frac{(1-Z_t)}{(1-Z_{t+1})} \right\} / Z_t = 1 + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

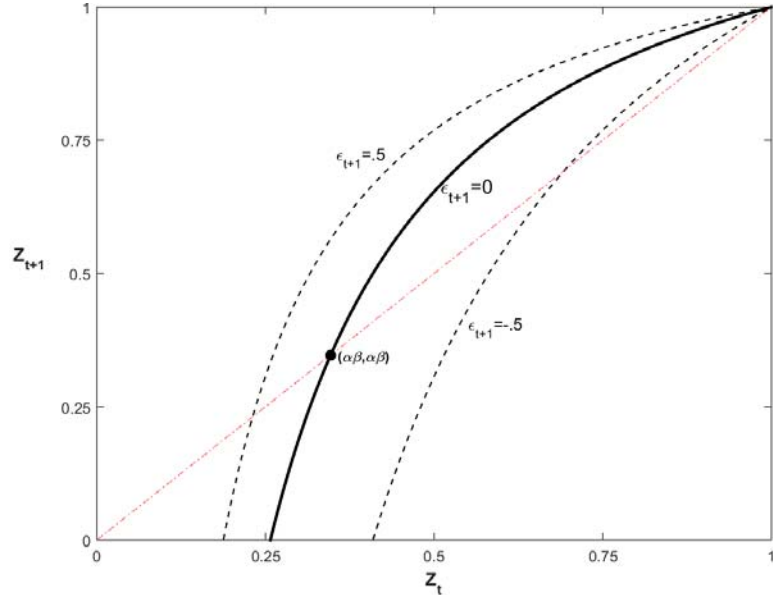
$$\Rightarrow Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1) / (1 + \varepsilon_{t+1}).$$



**Fig.1. Long & Plosser model: investment/output ratio at  $t+1, Z_{t+1}$ , as function of  $Z_t$  for  $\epsilon_{t+1} \in \{-0.5; 0; 0.5\}$**

$$Z_{t+1} = \Lambda(Z_t, \epsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1 + \epsilon_{t+1}); \quad \alpha = 0.35, \beta = 0.99.$$





$$Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1 + \varepsilon_{t+1})$$

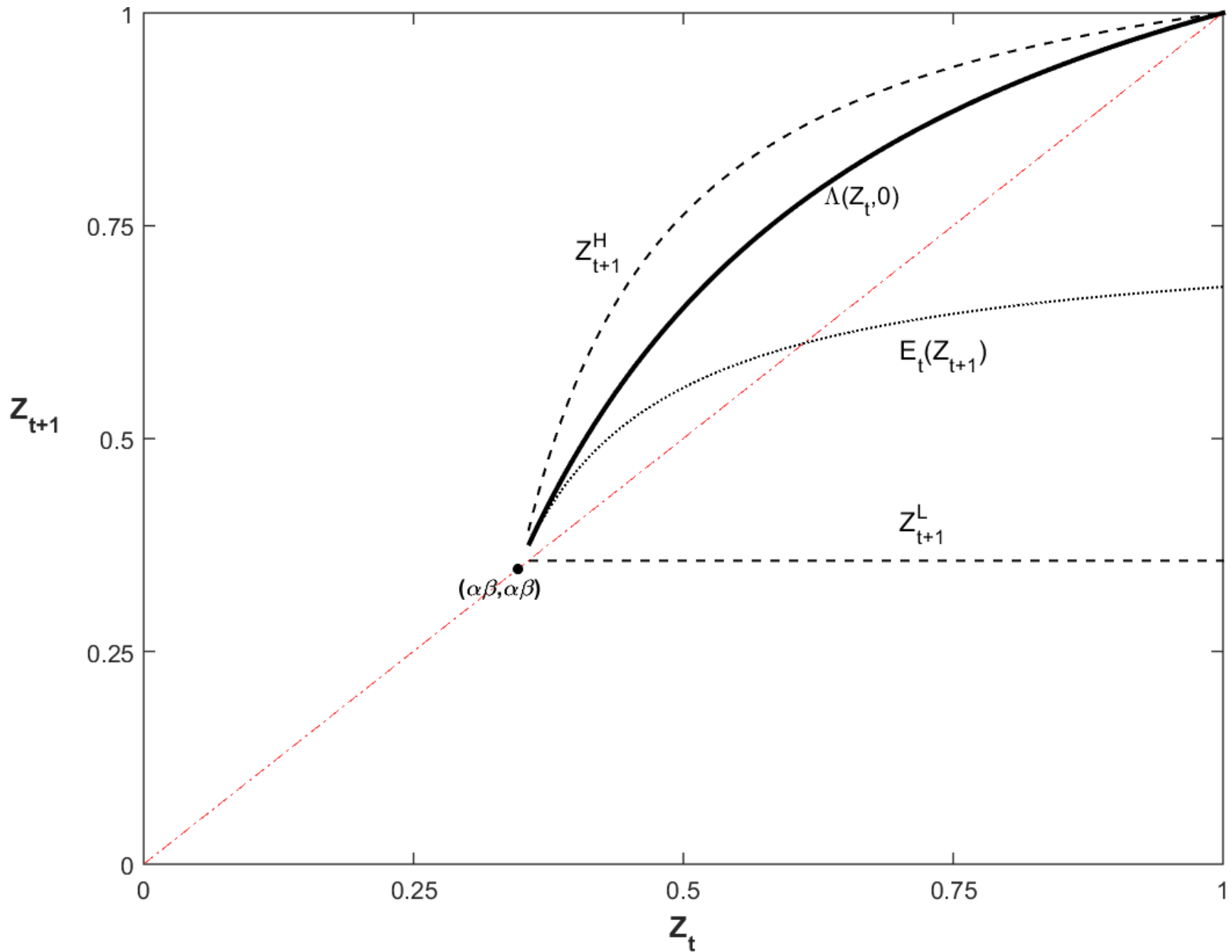
- When  $Z_t < \alpha\beta$ , the model can hit zero-capital corner solution in later periods  $\Rightarrow$  restrict attention to solutions with  $Z_\tau \in [\alpha\beta, 1) \quad \forall \tau$
- Support of  $\varepsilon_{t+1}$  has to be bounded below:  $\varepsilon_{t+1} \geq -1 + [\alpha\beta/(1-\alpha\beta)] \cdot [1/Z_t - 1]$   
 $\Rightarrow$  distribution of  $\varepsilon_{t+1}$  must depend on  $Z_t$  !
- Let  $\varepsilon_{t+1}$  only takes two values:  $-\bar{\varepsilon}_t$  and  $\bar{\varepsilon}_t \cdot \pi_t / (1 - \pi_t)$  with probabilities  $\pi_t$  and  $1 - \pi_t$ , respectively,  $\bar{\varepsilon}_t \in [0, 1]$ .  $\Rightarrow Z_{t+1}$  takes two values:  
 $Z_{t+1}^L \equiv \Lambda(Z_t, -\bar{\varepsilon}_t)$  &  $Z_{t+1}^H \equiv \Lambda(Z_t, \bar{\varepsilon}_t \pi_t / (1 - \pi_t))$  with  $Z_{t+1}^L \leq Z_{t+1}^H \leq 1$ .
- Postulate  $Z_{t+1}^L = f(Z_t)$ , with  $\alpha\beta \leq f(Z_t) \leq \Lambda(Z_t, 0)$  for  $Z_t \in [\alpha\beta, 1)$ .  
Solve  $Z_{t+1}^L \equiv \Lambda(Z_t, -\bar{\varepsilon}_t)$  for  $\bar{\varepsilon}_t$  & substitute into  $Z_{t+1}^H \equiv \Lambda(Z_t, \bar{\varepsilon}_t \pi_t / (1 - \pi_t))$

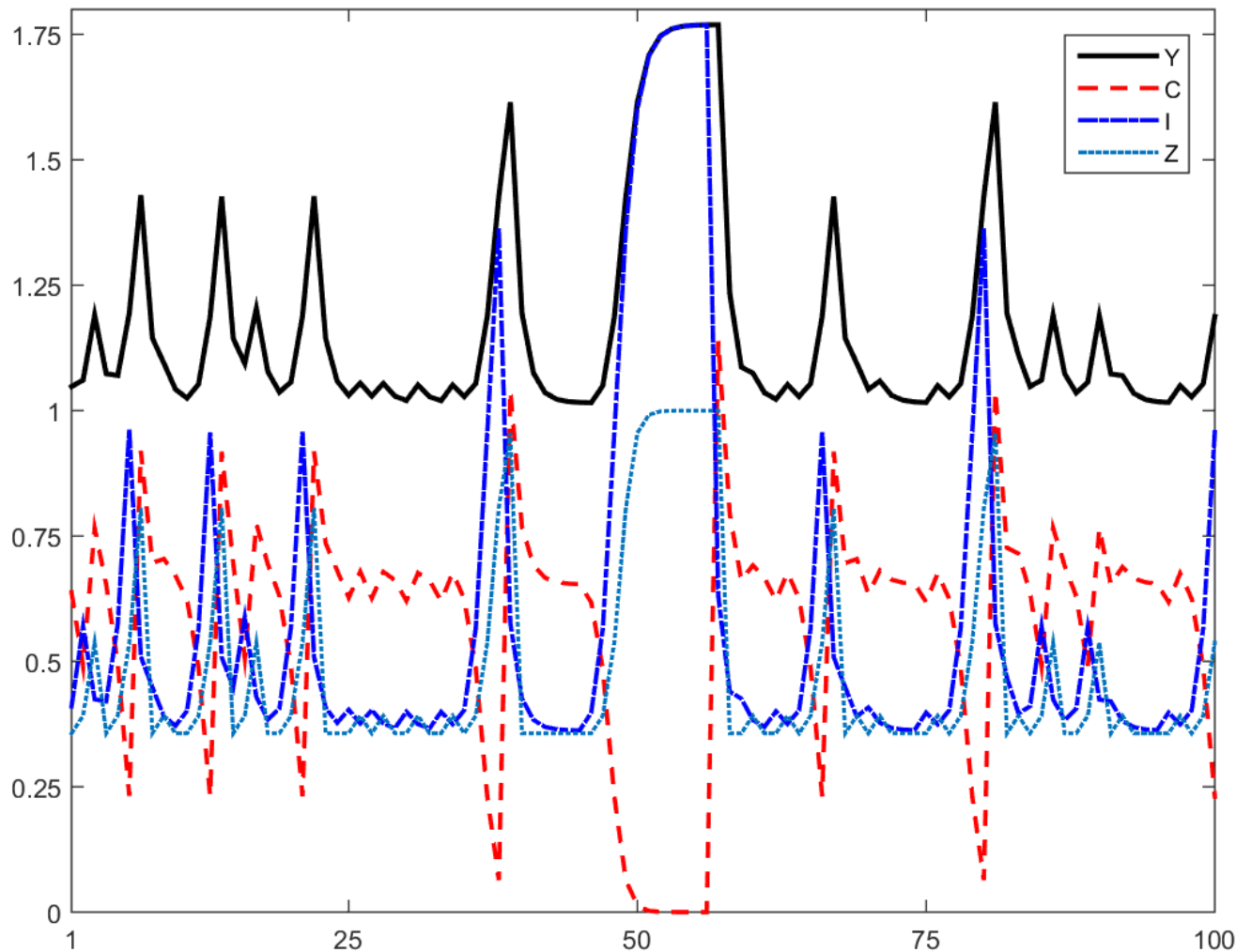
Two degrees of freedom in modeling sunspot:

- bust investment/GDP ratio,  $Z_{t+1}^L$
- conditional probability of bust,  $\pi_t$

# Specification I: $Z_{t+1}^L = \alpha\beta + \Delta$ , $\Delta = 0.01$ , $\pi = 0.5$

(When  $\Delta = 0$ , then  $Z = \alpha\beta = 0.346$  is absorbing state; thus set  $\Delta > 0$ )

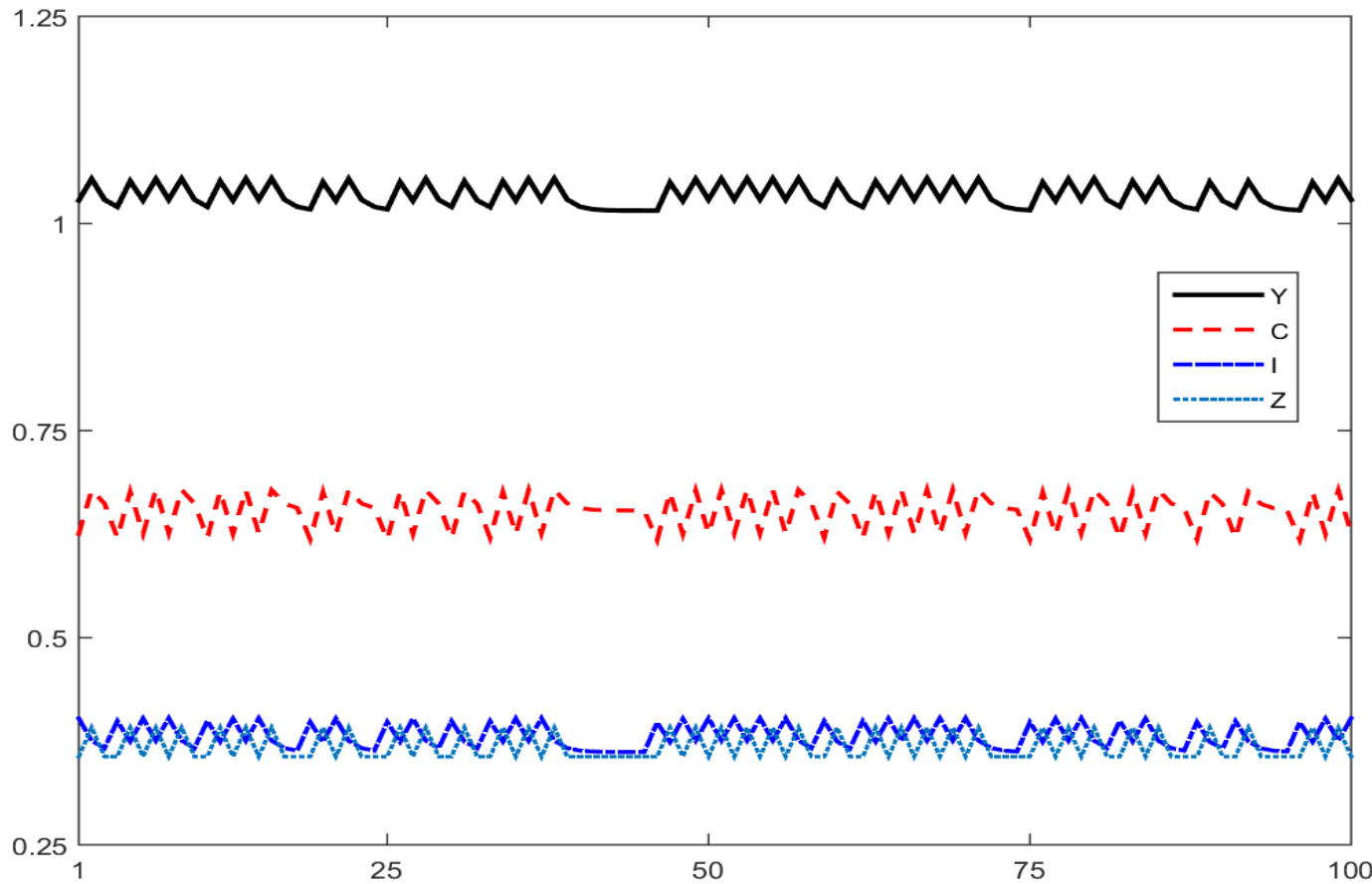




## Simulated series with constant probability: $\pi_t=0.5$

Simulated output (Y), consumption (C) and investment (I) normalized by steady state output

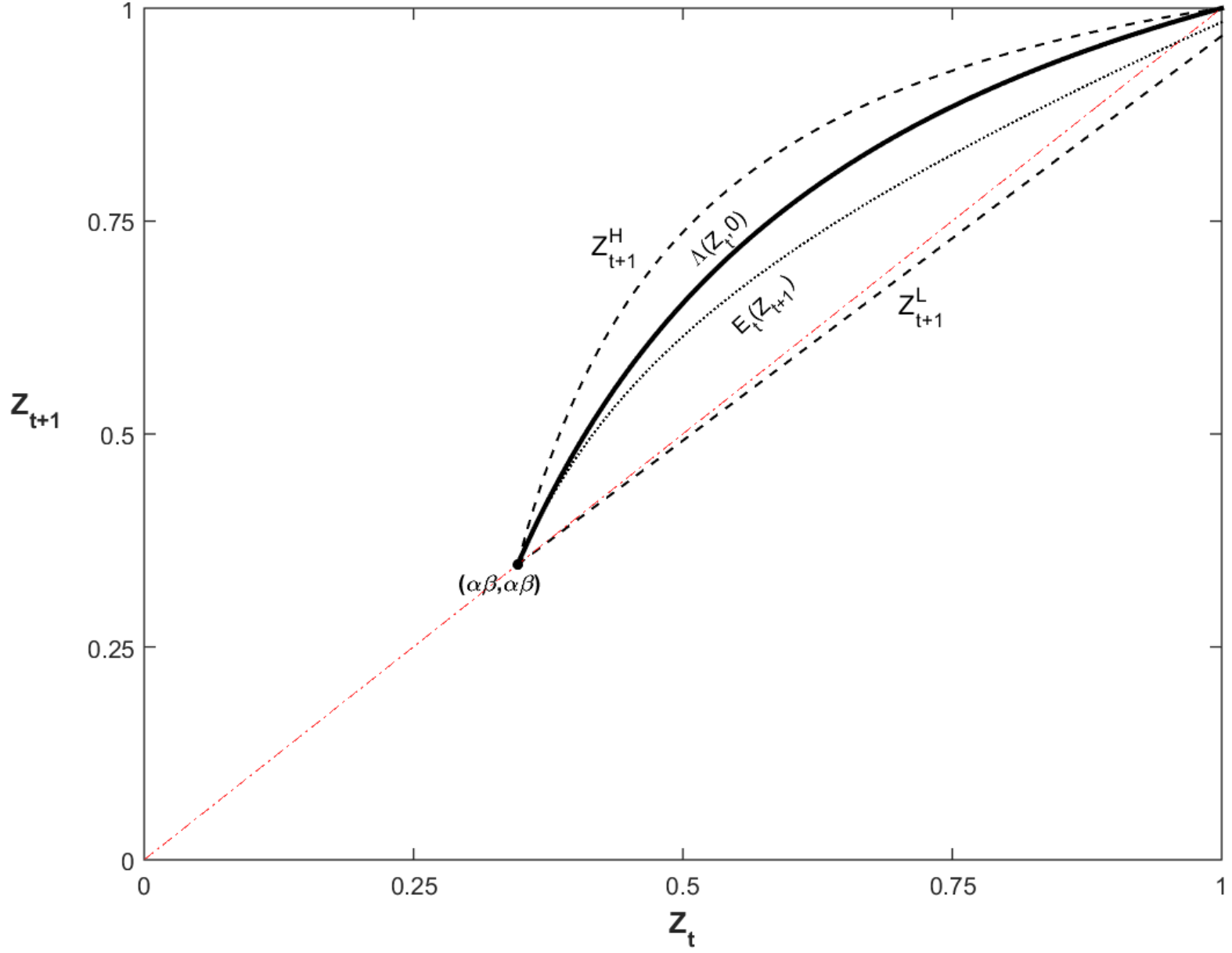
**Lower volatility if probability of investment bust rises once investment/output ratio  $Z_t$  crosses threshold.**

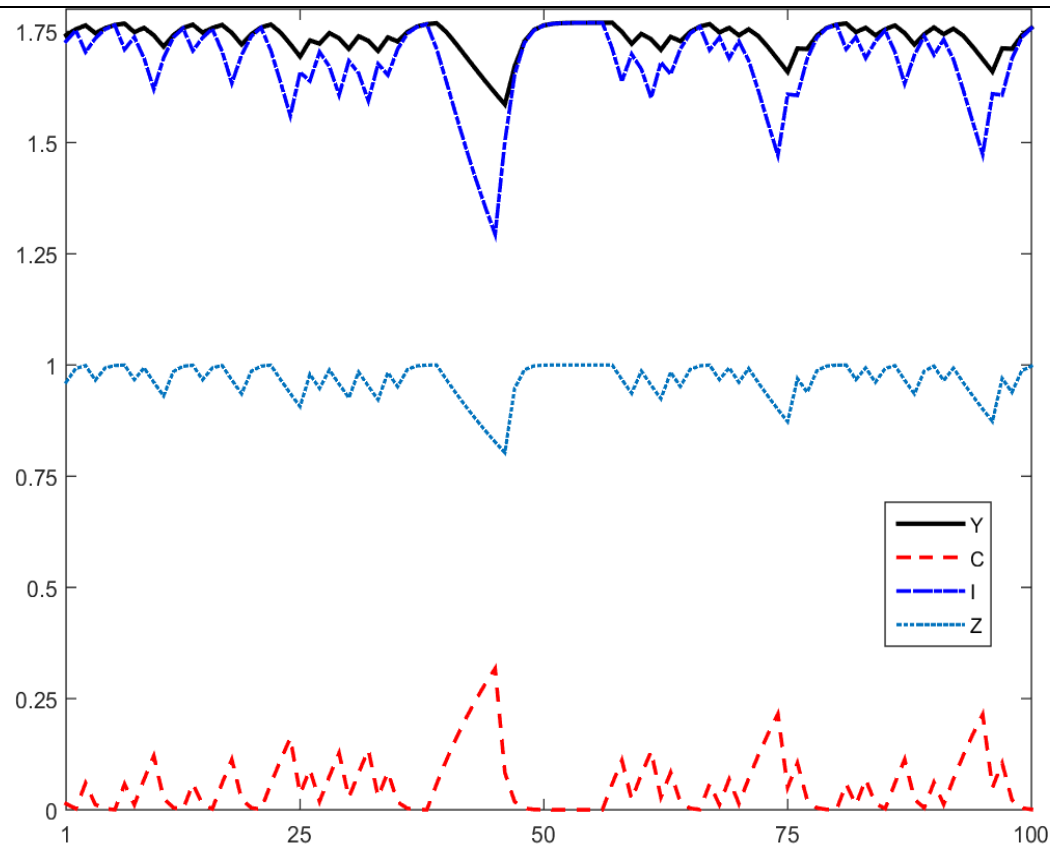


**Simulated series with state-contingent probability of bust:**  
 $\pi_t = 0.5$  for  $\alpha\beta + \Delta = 0.356 \leq Z_t \leq 0.36$  &  $\pi_t = 1 - 10^{-100}$  for  $Z_t > 0.36$

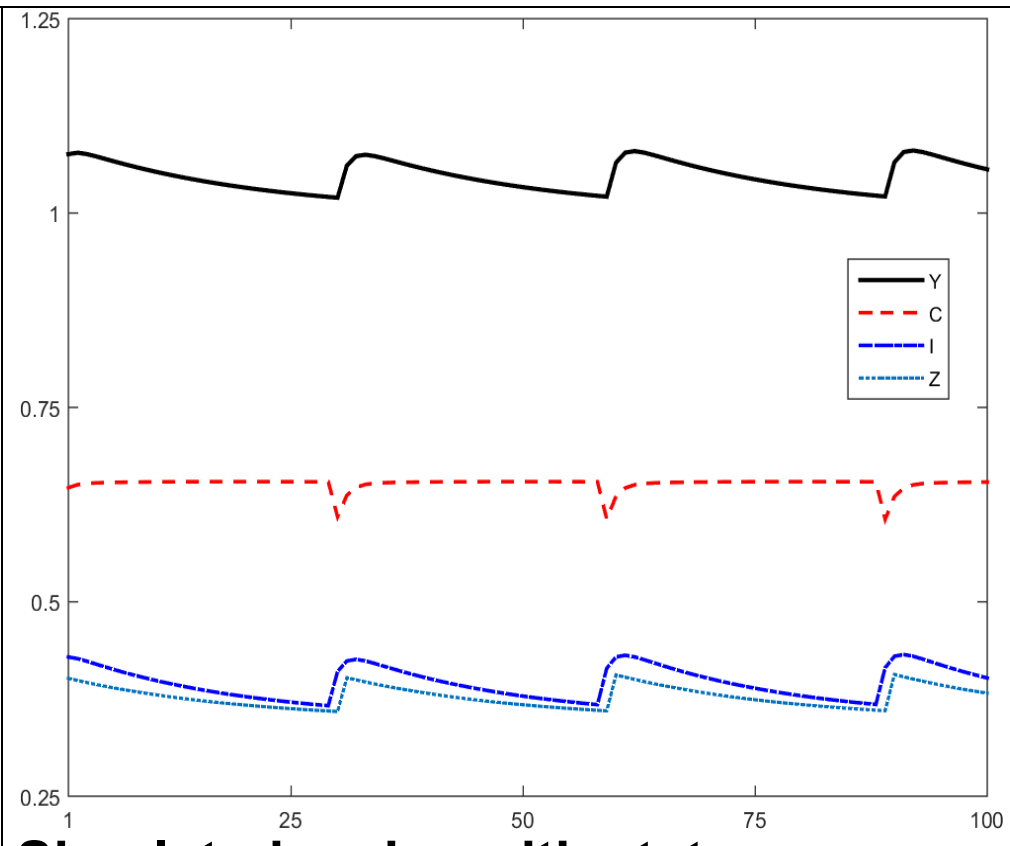
# Specification II: gradual contractions

$$Z_{t+1}^L = \alpha\beta + 0.95 \cdot (Z_t - \alpha\beta), \quad \pi = 0.5$$





**Simulated series with constant probability of bust:  $\pi_t=0.5$**



**Simulated series with state-contingent probability of bust:  
 $\pi_t=0.5$  for  $\alpha\beta+\Delta=0.356\leq Z_t\leq 0.36$  &  
 $\pi_t\approx 1$  for  $Z_t>0.36$**

**Table 1. Long-Plosser model with bubbles: predicted business cycle statistics**

	<u>Standard dev. %</u>			<u>Corr. with Y</u>		<u>Autocorr.</u>			<u>Mean (% deviation from SS)</u>			
	Y	C	I	C	I	Y	C	I	Y	C	I	Z
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
<b>(a) Specification I: <math>Z_t^L = a\beta + \Delta</math></b>												
$\pi_t = 0.5$	11.72	100.19	33.48	-0.42	0.62	0.62	0.47	0.62	13.49	-7.62	53.31	31.15
$\pi_t \approx 1$ for $z_t > 0.36$	1.33	3.51	3.82	0.77	-0.26	-0.26	-0.66	-0.26	3.27	-0.13	9.71	6.25
<b>(b) Specification II: <math>Z_t^L = a\beta + 0.95 \times (z_t - a\beta)</math></b>												
$\pi_t = 0.5$	1.73	210.28	4.94	-0.31	0.68	0.68	0.46	0.68	73.09	-89.68	380.11	177.21
$\pi_t \approx 1$ for $z_t > 0.36$	1.40	1.30	4.00	0.14	0.85	0.85	0.28	0.85	4.45	-0.26	13.34	8.46
<b>(c) US Data (from King and Rebelo (1999))</b>												
	1.81	1.35	5.30	0.88	0.80	0.88	0.80	0.87				



# Example II: RBC model with incomplete capital depreciation & endogenous labor

Max  $E_0 \sum_{t=0}^{\infty} \beta^s u(C_t, L_t)$  subject to resource constraint

$$C_t + I_t = Y_t \text{ with } I_t = K_{t+1} - (1 - \delta)K_t \text{ and } Y_t = (K_t)^\alpha (L_t)^{1-\alpha}.$$

$$u(C_t, L_t) = (1 - \sigma)^{-1} (C_t - v(L_t))^{1-\sigma}, v(L_t) = (\Psi / (1 + 1/\eta)) \{L_t^{1+1/\eta} - L^{1+1/\eta}\},$$

$$(1 - \alpha)Y_t / L_t = \Psi \cdot (L_t)^{1/\eta} \Rightarrow L_t = n(K_t) \equiv ((1 - \alpha)(K_t^\alpha) / \Psi)^{1/(\alpha + 1/\eta)}$$

$$E_t \beta \{ (C_{t+1} - v(L_{t+1}))^{-\sigma} / (C_t - v(L_t))^{-\sigma} \} (\alpha Y_{t+1} / K_{t+1} + 1 - \delta) = 1$$

$$\beta \{ (C_{t+1} - v(L_{t+1}))^{-\sigma} / (C_t - v(L_t))^{-\sigma} \} (\alpha Y_{t+1} / K_{t+1} + 1 - \delta) = 1 + \varepsilon_{t+1},$$

$$\varepsilon_{t+1} : \text{sunspot with } E_t \varepsilon_{t+1} = 0$$

$$\Rightarrow K_{t+2} = \kappa(K_{t+1}, K_t, \varepsilon_{t+1})$$

# Intuition about stationary sunspot equilibrium:

- $K_{t+2}^L = \lambda(K_{t+1})$ : stationary no-sunspot decision rule
- Gap between  $K$  with & without sunspot:  
 $g_{t+1} \equiv K_{t+2} - \lambda(K_{t+1})$ .

In linearized system: gap explodes

$$g_{t+1} = (\kappa_1 - \lambda') \cdot g_t + \kappa_3 \cdot \varepsilon_{t+1}; \quad (\kappa_1 - \lambda') > 1$$

**In non-linear system: gap can be stationary**  
**Non-sunspot solution is 'attractor'**

# $K_{t+2} = \kappa(K_{t+1}, K_t, \varepsilon_{t+1})$ : solution with sunspots

■ Assume  $\varepsilon_{t+1}$  takes only two values,  $-\bar{\varepsilon}_t$  and  $\bar{\varepsilon}_t\pi/(1-\pi)$ , with probabilities  $\pi$  &  $1-\pi$ ;  $\bar{\varepsilon}_t \in [0,1]$ .

● **‘Bust’**:  $K_{t+2}^L = \lambda(K_{t+1})$     **no-sunspot decision rule**

$K_{t+2}^L = \kappa(K_{t+1}, K_t, -\bar{\varepsilon}_t) \Rightarrow$  can solve this for  $\bar{\varepsilon}_t$

● This pins down **‘Boom’** capital stock:

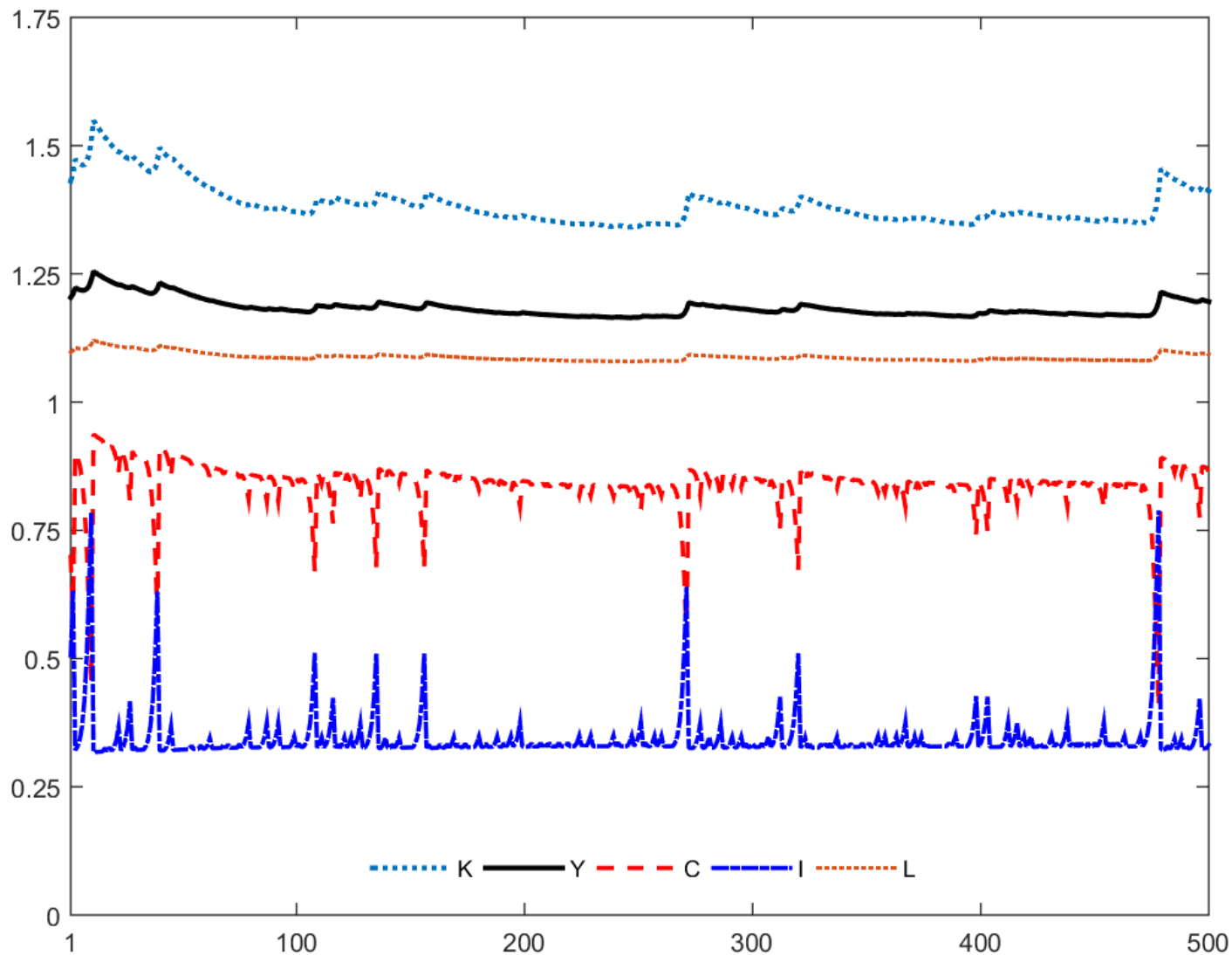
$K_{t+2}^H = \kappa(K_{t+1}, K_t, \bar{\varepsilon}_t\pi/(1-\pi))$

Parameters:

$\beta=0.99$ ,  $\alpha=0.35$ ,  $\delta=0.025$  (depreciation rate)

$\sigma=\eta=1$  (risk aversion, labor supply elasticity)

$\pi = 0.5$  (bust probability)



**Non-linear RBC model (incomplete capital depreciation, variable labor) with bubbles.** Simulated paths of GDP (Y), consumption (C), investment (I) are normalized by steady state GDP. Capital (K) and hours series (L) are normalized by their respective steady states.

**Table 2. RBC model (incomplete capital depreciation) with bubbles: predicted business cycle statistics**

	<u>Standard dev. %</u>				<u>Corr. with Y</u>			<u>Autocorr.</u>				<u>Mean (% deviation from SS)</u>				
	<i>Y</i>	<i>C</i>	<i>I</i>	<i>L</i>	<i>C</i>	<i>I</i>	<i>L</i>	<i>Y</i>	<i>C</i>	<i>I</i>	<i>L</i>	<i>Y</i>	<i>C</i>	<i>I</i>	<i>L</i>	<i>K</i>
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
$\pi_t=0.5$	0.46	9.67	11.12	0.23	0.13	-0.21	1.00	0.91	0.54	0.52	0.91	18.6	11.8	39.0	8.90	39.0

## Example III: Small Open Economy

$$\text{Max } E_0 \sum_{t=0}^{\infty} \beta^s \ln(C_t) \quad \text{s.t. } C_t + A_{t+1} = A_t \cdot R_t + Y$$

$C_t$ : consumption;  $Y$ : output

$A_{t+1}$ : net foreign assets, NFA;  $R_t$ : gross interest rate;

- Euler equation:  $\beta E_t (C_{t+1}/C_t)^{-1} R_{t+1} = 1$

- $R_t = R(A_t)$ , with  $R' < 0$

- No-sunspots solution: policy function  $A_{t+1} = \lambda(A_t)$

## ■ How to construct stationary sunspot equilibrium:

● Euler equation  $\Rightarrow \beta(C_{t+1}/C_t)^{-1}R(A_{t+1})=1+\varepsilon_{t+1}$ ;

$\varepsilon_{t+1}$ : sunspot with  $E_t \varepsilon_{t+1}=0$ .

$$\Rightarrow C_{t+1} = C_t \cdot \beta \cdot R(A_{t+1}) / (1 + \varepsilon_{t+1})$$

● Budget constraint:  $C_t = A_t \cdot R(A_t) + Y - A_{t+1}$

● Substitute budget constraint into Euler equation:

$$\Rightarrow A_{t+2} = A_{t+1}R(A_{t+1}) + Y - (A_tR(A_t) + Y - A_{t+1}) \cdot \{\beta R(A_t) / (1 + \varepsilon_{t+1})\}^{1/\sigma}$$

$A_{t+2} = \kappa(A_{t+1}, A_t, \varepsilon_{t+1})$ : NFA law of motion with sunspot

$\kappa_\varepsilon > 0$ ,  $\kappa_{\varepsilon\varepsilon} > 0$ : non-linearity permits stationary sunspot equil.

$A_{t+2} = \kappa(A_{t+1}, A_t, \varepsilon_{t+1})$ : NFA law of motion with sunspot

$A_{t+2} = \lambda(A_{t+1})$ : stationary no-sunspots decision rule

■ **Structure of stationary sunspot equilibrium:**

● Gap between NFA with & without sunspot:

$g_{t+1} \equiv A_{t+2} - \lambda(A_{t+1})$ . In linearized system: gap explodes

$$g_{t+1} = (\kappa_1 - \lambda') \cdot g_t + \kappa_3 \cdot \varepsilon_{t+1}; \quad (\kappa_1 - \lambda') > 1$$

In non-linear model: gap can be stationary



# $A_{t+2} = \kappa(A_{t+1}, A_t, \varepsilon_{t+1})$ : NFA law of motion with sunspot

► Assume  $\varepsilon_{t+1}$  takes only two values,  $-\bar{\varepsilon}_t$  and  $\bar{\varepsilon}_t \pi_t / (1 - \pi_t)$ , with probabilities  $\pi_t$  &  $1 - \pi_t$ ;  $\bar{\varepsilon}_t \in [0, 1]$ .

● ‘Bust’:  $A_{t+2}^L = \lambda(A_{t+1}) + \Delta$ . **No-sunspot decision rule.**  $\Delta > 0$

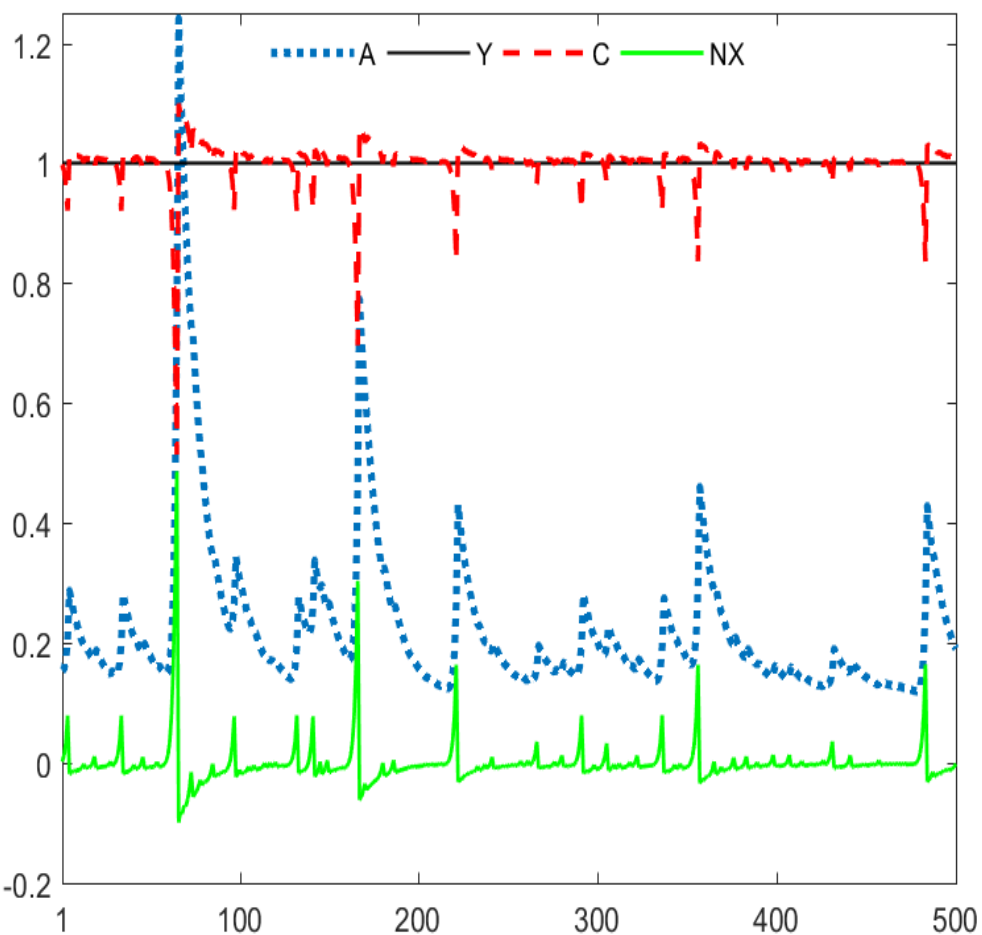
$A_{t+2}^L = \kappa(A_{t+1}, A_t, -\bar{\varepsilon}_t)$ : this pins down  $\bar{\varepsilon}_t$

● ‘Boom’:  $A_{t+2}^H = \kappa(A_{t+1}, A_t, \bar{\varepsilon}_t \pi_t / (1 - \pi_t))$

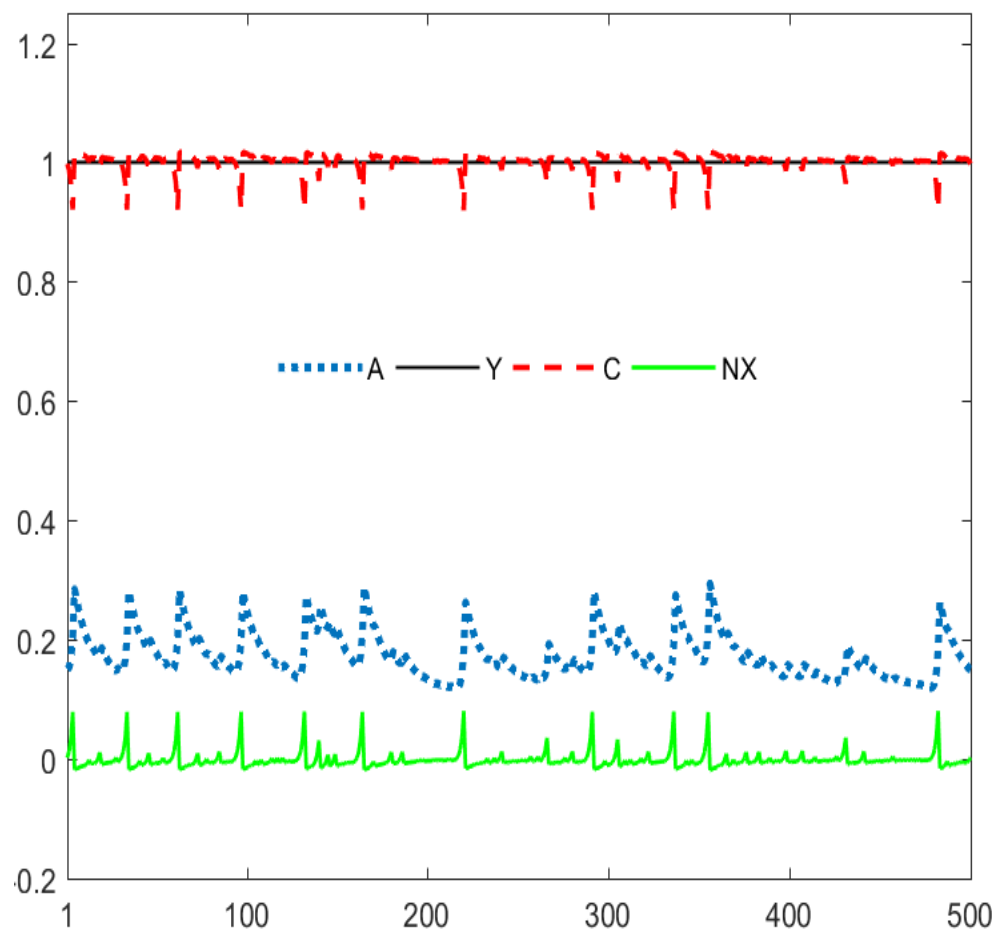
Parameters:

$\beta = 0.99$ ;  $R(A_t) = \exp(-a \cdot A_t / Y) / \beta$ ,  $a = 0.01$ .

$\pi_t = 0.5$  (bust probability)



**(a) Constant bust probability:  $\pi_t=0.5$**



**(b) State-contingent bust probability:**  
 $\pi_t=0.5$  for  $A_{t+1}/Y \leq 0.25$ ;  
 $\pi_t \approx 1$  for  $A_{t+1}/Y > 0.25$

# Non-linear Small Open Economy model with bubbles

Simulated paths of net foreign assets (A), GDP (Y), consumption (C) and net exports (NX). All series normalized by steady state GDP.

**Table 3. Small Open Economy model (endowments) with bubbles: predicted business cycle statistics**

	<u>Standard dev. %</u>			<u>Corr. with Y</u>		<u>Autocorr.</u>			<u>Mean (% deviation from SS)</u>			
	A	C	NX	C	NX	A	C	NX	A	Y	C	NX
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
$\pi_t=0.5$	11.43	9.07	4.95	---	---	0.90	0.53	0.52	24.44	0.00	0.09	-0.09
$\pi_t=1$ for $A_{t+1}/y > 0.25$	2.07	1.30	1.27	---	---	0.81	0.26	0.26	16.70	0.00	0.13	-0.13

# CONCLUSIONS

- Stationary sunspot equilibria exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions.
- In the sunspot equilibria considered here, the economy temporarily diverges from the no-sunspots trajectory, before abruptly reverting towards that trajectory.
- In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity.

# ADDITIONAL MATERIAL

Blanchard (1979):

$$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1 \quad \Rightarrow \quad y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

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How non-linearity may generate stationary bubble:

Assume:  $E_t \exp(z_{t+1} - \lambda z_t) = a$ ,  $\lambda > 1$ ,  $a > 0$

$$\Rightarrow \exp(z_{t+1} - \lambda z_t) = a + \eta_{t+1} \quad \text{with} \quad E_t \eta_{t+1} = 0$$

$$\Rightarrow z_{t+1} = \lambda z_t + \log(a + \eta_{t+1}). \quad \text{Let } y_t \equiv z_t + \ln(a)/(\lambda - 1), \quad \varepsilon_{t+1} \equiv \eta_{t+1}/a$$

$$\Rightarrow y_{t+1} = \lambda \cdot y_t + \ln(1 + \varepsilon_{t+1}), \quad E_t \varepsilon_{t+1} = 0$$

$y_{t+1}$  is **concave** in  $\varepsilon_{t+1} \Rightarrow E_t y_{t+1} < \lambda \cdot y_t$

Let  $\varepsilon_{t+1} \in \{-\bar{\varepsilon}_t; \bar{\varepsilon}_t \pi / (1 - \pi)\}$  with prob.  $\pi, 1 - \pi$ .  $\bar{\varepsilon}_t > 0$

Set  $\bar{\varepsilon}_t \in [0, 1)$  so that  $y_{t+1} = \lambda \cdot y_t + \ln(1 - \bar{\varepsilon}_t) = \Delta$

$$y_{t+1} = y_{t+1}^H \equiv \lambda \cdot y_t + \ln\{1 + [1 - \exp(\Delta - \lambda \cdot y_t)] \cdot \pi / (1 - \pi)\} \quad \text{with prob. } 1 - \pi$$

$$y_{t+1} = \Delta \quad \text{with probability } \pi$$