Rational Bubbles in Non-Linear Business Cycle Models

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April 9, 2019

Main result:

non-linear DSGE models have more stationary equilibria than you think!

Blanchard & Kahn (1980): conditions for existence of unique **stationary** solution of **linear(ized)** models are IRRELEVANT for non-linear models

This paper shows: stationary sunspot equilibria exist in standard non-linear DSGE models, <u>even when the linearized versions of these models have unique solution</u>

$\Rightarrow$ **DSGE models may have multiple equilibria, if non-linearity taken into account**

These sunspot equilibria look like ‘bubbles’: Economy temporarily diverges from no-sunspots trajectory, before reverting abruptly towards no-sunspots trajectory

“Divergent behavior”: similar to ‘rational’ bubbles in linear models (Blanchard (1979))

Big difference compared to Blanchard bubbles: bubbles here are STATIONARY.

$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1$

Unique stable solution: $y_t = 0$

Blanchard (1979) :

Bubble: $y_{t+1} = (\lambda/(1-\pi)) \cdot y_t$ with probability $1-\pi$

$y_{t+1} = 0$ with probability π

$\lim_{s\to\infty} E_t y_{t+s} = \pm\infty$

expected path of bubble diverges to ±∞

Expected path of bubbles in <u>non-linear</u> DSGE described here do NOT diverge to ±∞

Note: Can construct DSGE models whose linearized versions have sunspots:

$E_t y_{t+1} = \lambda \cdot y_t$ **need** $|\lambda| < 1$**.** $\Rightarrow y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}$ is stationary solution for <u>any</u> $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1} = 0$

Needed ingredients:

- Increasing returns, externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999))
- Financial frictions (e.g., Martin and Ventura (2018))
- Overlapping generations (e.g., Galí (2018))

Specific assumptions & calibrations that deliver $|\lambda| < 1$ can be debatable & fragile (e.g. in standard OLG model: need dynamic inefficiency, r≤g)

<u>By contrast, paper here argues that very standard DSGE model can deliver stationary sunspot equilibria, if non-linearities are considered.</u>

Basic intuition I:

Consider model with just 1 non-predetermined variable (no exogenous driver)

$$E_t G(Y_{t+1}, Y_t) = 0$$

Linearization (around steady state) gives:

$$E_t y_{t+1} = \lambda \cdot y_t, \quad y_t \equiv Y_t - Y^{SS}$$

Linearized model has unique non-explosive solution iff $|\lambda| > 1$. That unique solution is: $y_t = 0$

(Blanchard & Kahn (1980), Prop. 1)

I show: even when $|\lambda|>1$, the **non-linear** model can have stationary sunspot equilibrium

$$E_t G(Y_{t+1}, Y_t) = 0 \quad \Rightarrow \quad G(Y_{t+1}, Y_t) = \varepsilon_{t+1} \text{ for } E_t \varepsilon_{t+1} = 0$$

$\Rightarrow Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1})$. ε_{t+1}: "sunspot shock"

Paper shows: even if $|\Lambda_Y| > 1$, there may exist a process $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1} = 0$ such that $\{Y_{t+1}\}$ is stationary.

Note: when white noise $\{\varepsilon_{t+1}\}$ is fed into $Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1})$, then $\{Y_{t+1}\}$ diverges if $|\Lambda_Y| > 1$.

Key requirements for stationary solution:

- $Y_{t+1}=\Lambda(Y_t,\varepsilon_{t+1})$ has to be NON-LINEAR in ε_{t+1}
- Distribution of ε_{t+1} has to depend on Y_t

$$Y_{t+1}\cong\Lambda(Y_t,0)+\Lambda_\varepsilon(Y_t,0)\cdot\varepsilon_{t+1}+\tfrac{1}{2}\Lambda_{\varepsilon\varepsilon}(Y_t,0)\cdot(\varepsilon_{t+1})^2$$

$$E_tY_{t+1}\cong\Lambda(Y_t,0)+\tfrac{1}{2}\Lambda_{\varepsilon\varepsilon}(Y_t,0)\cdot E_t(\varepsilon_{t+1})^2$$

Let $E_t(\varepsilon_{t+1})^2=f(Y_t)\geq 0$. If $\Lambda_{\varepsilon\varepsilon}(Y_t,0)\neq 0$ then can set $E_t(\varepsilon_{t+1})^2=f(Y_t)$ such that $|dE_tY_{t+1}/dY_t|<1$:
"MEAN REVERSION"

Example: $\Lambda_Y(Y_t,0)>1$, $\Lambda_{\varepsilon\varepsilon}(Y_t,0)<0$. Then need $f'(Y_t)>0$ for mean reversion: $E_t(\varepsilon_{t+1})^2$ must be increasing in Y_t.

Basic intuition II: RBC model

$C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t), \ F' > 0, \ F'' < 0$

$\beta\{[E_t u'(C_{t+1})]/u'(C_t)\} \cdot F'(K_{t+1}) = 1$; assume $u''' > 0$ (CRRA)

Sunspot: assume $K_{t+1} \uparrow \Rightarrow C_t \downarrow \ u'(C_t) \uparrow, F'(K_{t+1}) \downarrow$

Euler eqn requires: $E_t u'(C_{t+1}) = E_t u'(F(K_{t+1}) - K_{t+2}) \uparrow$

- In <u>deterministic</u> economy: need $C_{t+1} \downarrow$ & $K_{t+2} \uparrow$

K_{t+2} has to rise <u>**more**</u> than K_{t+1} ! $\Rightarrow$ <u>**K diverges**</u>

- With stochastic sunspot: K_{t+2} random.

$u'(C_{t+1})$ is convex in $K_{t+2} \Rightarrow$ if $Var_t(K_{t+2})$ rises,

$E_t K_{t+2}$ may rise less than K_{t+1}:

$\Rightarrow$ possibility of **mean reversion**

Several of the models considered below are usually presented as outcome of decision problem of an infinitely-lived representative agent.

Bubbles violate the transversality condition (TVC) of infinitely lived household:

$$\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1} > 0$$

This paper disregards TVC:

1) Lansing (2010) disregards the TVC in a Lucas-style asset pricing models with bubbles, arguing that "agents are forward-looking but not to the extreme degree implied by the transversality condition"

2) In richer models with heterogeneous agents and distortions: equilibrium is not solution of decision problem of representative agent.

Detection of TVC violations in stochastic economies: virtually impossible, even with very long simulation runs (billions of periods):
States with very low consumption might only occur with extremely small probabilities.

3) Assume OLG population structure with agents who live $N<\infty$ periods: then the TVC (infinite horizon) does not hold

Novel result about OLG economy:
(i) if there is a complete financial market that allows all generations alive at both dates *t* **and** *t*+1
(ii) if the generation born at date *t* receives a wealth endowment that is a constant share 1/N of aggregate date *t* wealth (across all generations)

THEN

an 'aggregate' Euler equation holds that is identical to the Euler equation of a representative infinitely lived household:

$$\beta E_t\{u'(C_{t+1})/u'(C_t)\}MPK_{t+1}=1$$

BUT: there is no TVC in the OLG economy!

OLG structure with efficient intergenerational risk sharing: justification for considering macro models that lack a TVC, but whose other equilibrium conditions are identical to those of standard business cycle models (that assume infinitely lived agents)

Detailed Example I:
Long-Plosser RBC model with sunspots

$u(C)=\ln(C)$; $C_t+K_{t+1}=Y_t$; $Y_t=F(K_t)\equiv\theta\cdot(K_t)^{\alpha}$, $0<\alpha<1$

Euler equation: $\beta E_t\{u'(C_{t+1})/u'(C_t)\}\cdot F'(K_{t+1})=1$

$\Rightarrow \beta E_t\{C_t/C_{t+1}\}\cdot\alpha Y_{t+1}/K_{t+1}=1$

$\Rightarrow \beta E_t\{(Y_t-K_{t+1})/(Y_{t+1}-K_{t+2})\}\cdot\alpha Y_{t+1}/K_{t+1}=1$

$\Rightarrow \alpha\beta\cdot E_t\{(1-K_{t+1}/Y_t)/(1-K_{t+2}/Y_{t+1})\}\cdot Y_t/K_{t+1}=1$

$\Rightarrow \alpha\beta\cdot E_t\{(1-Z_t)/(1-Z_{t+1})\}/Z_t=1$,

$Z_t\equiv K_{t+1}/Y_t$: investment/output ratio

Textbook solution: $Z_t=\alpha\beta$

$$\alpha\beta \cdot E_t\{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1$$

Linearization around $Z = \alpha\beta$:

$$E_t z_{t+1} = \lambda \cdot z_t, \quad z_t \equiv Z_t - Z; \quad \lambda \equiv 1/(\alpha\beta) > 1.$$

$\Rightarrow z_t = 0$ is unique non-explosive solution of linearized model.

But: **non-linear model has other stationary solutions.**

$$\alpha\beta \cdot \{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1 + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

$$\Rightarrow Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1+\varepsilon_{t+1}).$$

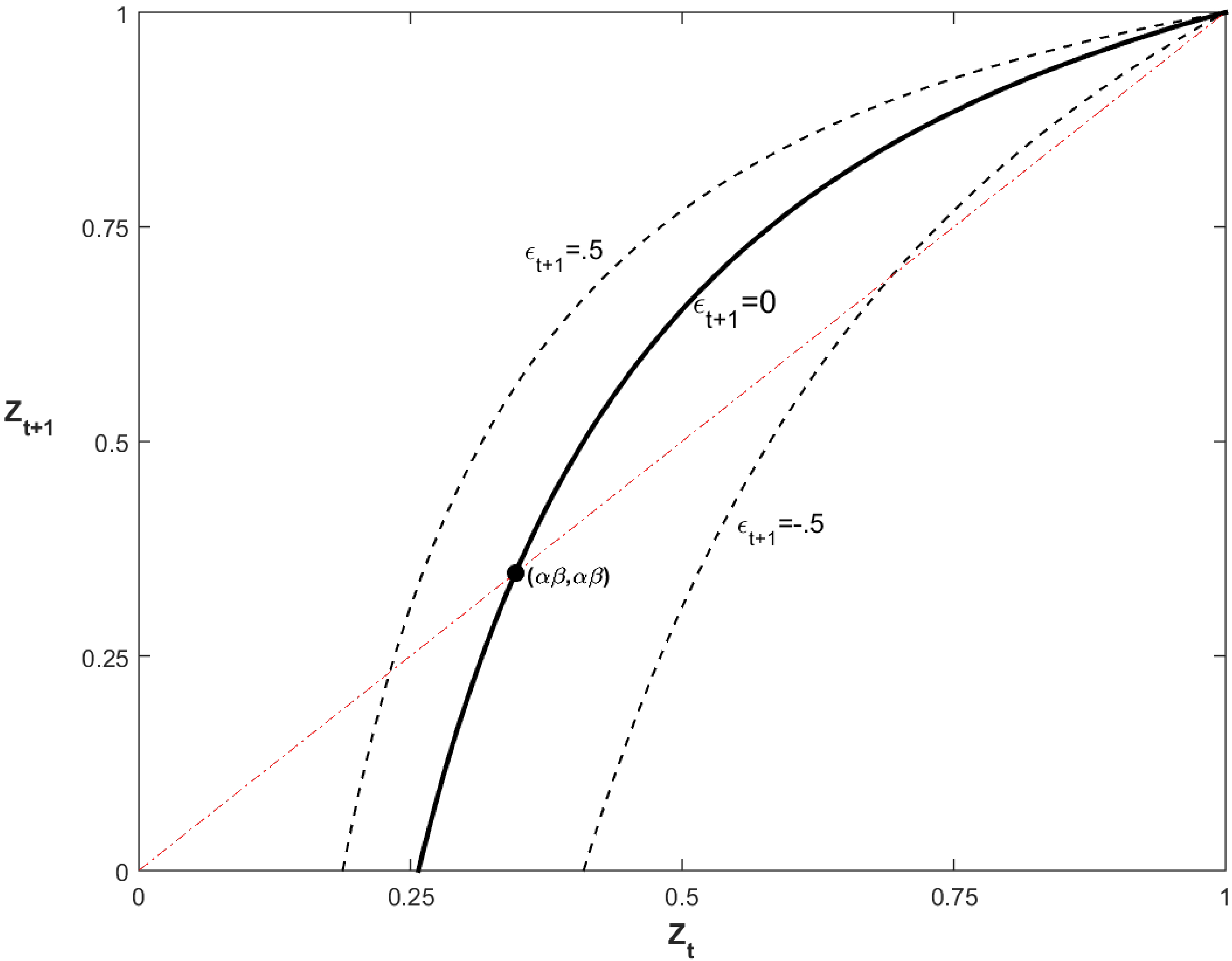
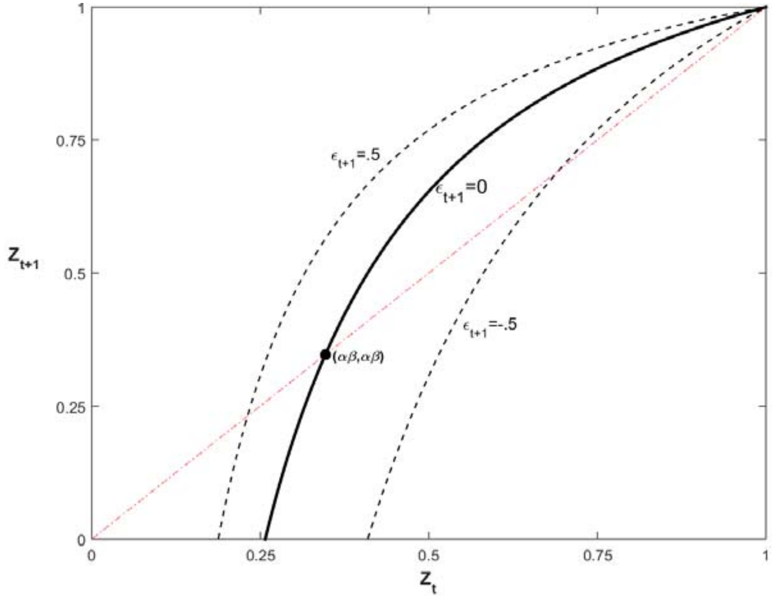


Fig.1. Long & Plosser model: investment/output ratio at t+1, Z_{t+1}**, as function of** Z_t **for** $\varepsilon_{t+1} \in \{-0.5;0;0.5\}$

$Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1}) \equiv 1-\alpha\beta(1/Z_t-1)/(1+\varepsilon_{t+1})$; $\alpha=0.35, \beta=0.99.$



$Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1})\equiv 1-\alpha\beta(1/Z_t-1)/(1+\varepsilon_{t+1})$

- When $Z_t<\alpha\beta$, the model can hit zero-capital corner solution in later periods $\Rightarrow$ restrict attention to solutions with $Z_\tau\in[\alpha\beta,1)\ \forall\tau$

- Support of ε_{t+1} has to be bounded below: $\varepsilon_{t+1}\geq -1+[\alpha\beta/(1-\alpha\beta)]\cdot[1/Z_t-1]$
$\Rightarrow$ distribution of ε_{t+1} must depend on Z_t !

- Let ε_{t+1} only takes two values: $-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\cdot\pi_t/(1-\pi_t)$ with probabilities π_t and $1-\pi_t$, respectively, $\overline{\varepsilon_t}\in[0,1]$. $\Rightarrow Z_{t+1}$ takes two values:

$$Z^L_{t+1}\equiv\Lambda(Z_t,-\overline{\varepsilon_t})\ \&\ Z^H_{t+1}\equiv\Lambda(Z_t,\overline{\varepsilon_t}\pi_t/(1-\pi_t))\text{ with }Z^L_{t+1}\leq Z^H_{t+1}\leq 1.$$

- Postulate $Z^L_{t+1}=f(Z_t)$, with $\alpha\beta\leq f(Z_t)\leq\Lambda(Z_t,0)$ for $Z_t\in[\alpha\beta,1)$.

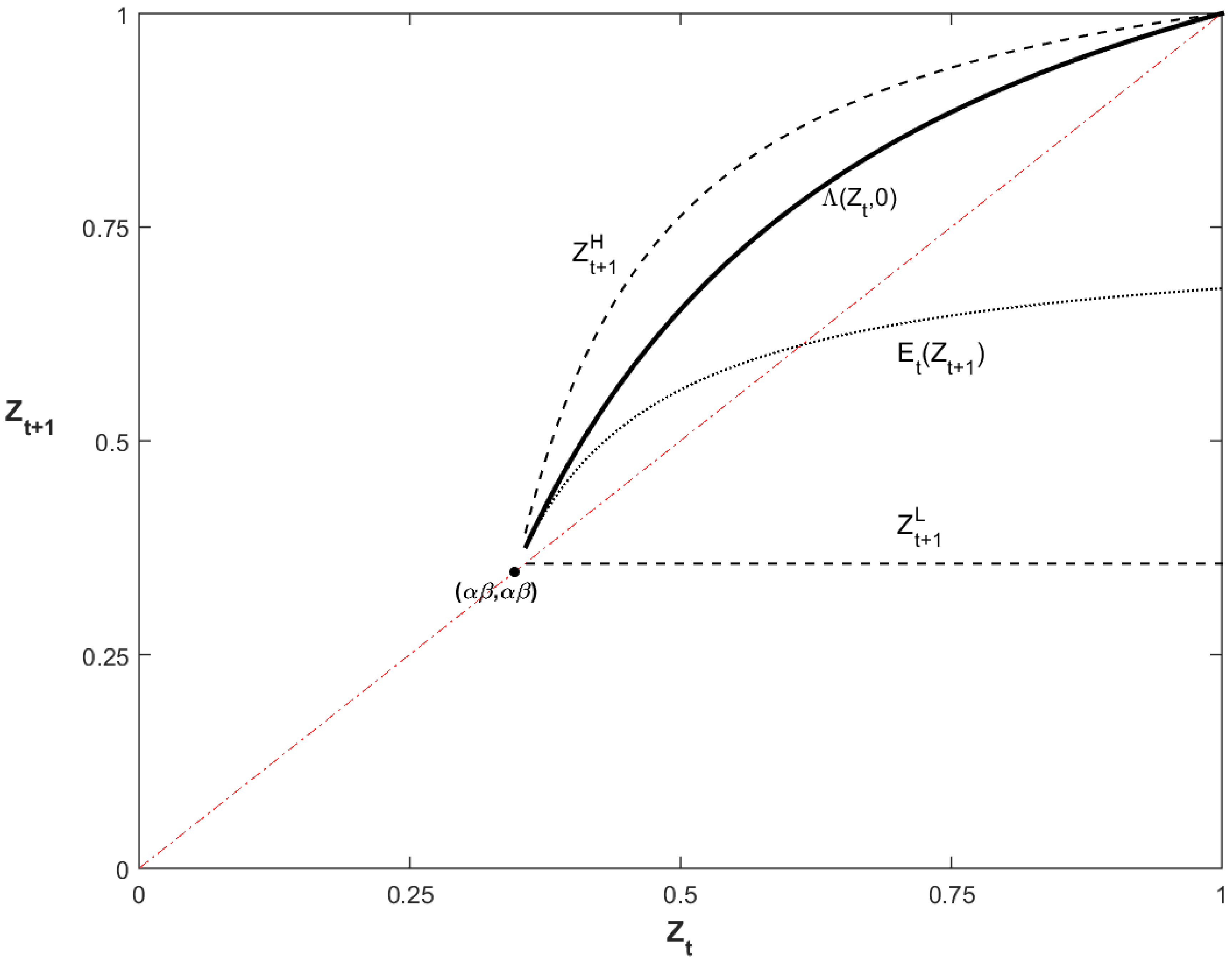
Solve $Z^L_{t+1}\equiv\Lambda(Z_t,-\overline{\varepsilon_t})$ for $\overline{\varepsilon_t}$ & substitute into $Z^H_{t+1}\equiv\Lambda(Z_t,\overline{\varepsilon_t}\pi_t/(1-\pi_t))$

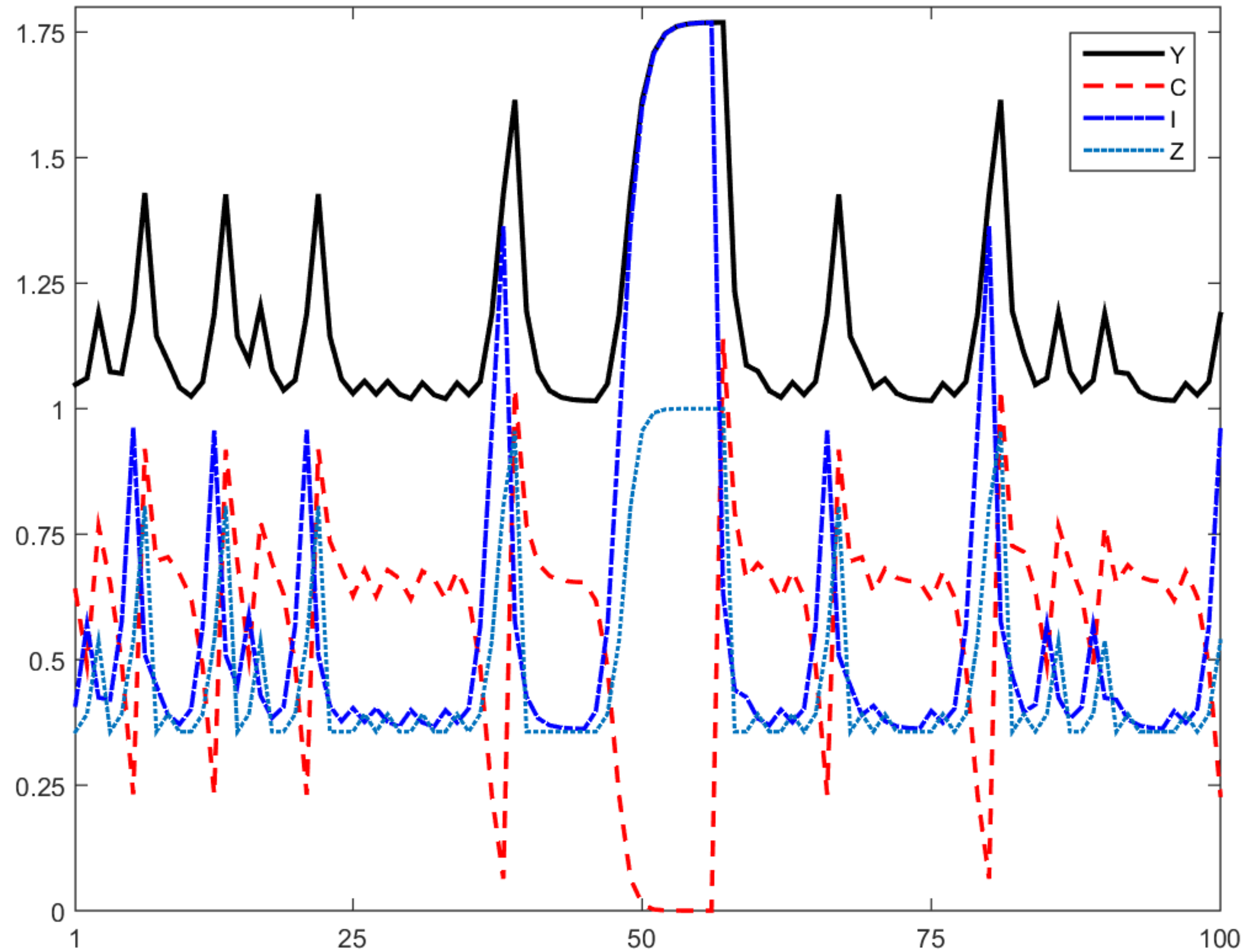
Two degrees of freedom in modeling sunspot:

- bust investment/GDP ratio, Z_{t+1}^{L}
- conditional probability of bust, π_t

Specification I: $Z^L_{t+1}=\alpha\beta+\Delta$, $\Delta=0.01$, $\pi=0.5$

(When $\Delta=0$, then $Z=\alpha\beta=0.346$ is absorbing state; thus set $\Delta>0$)

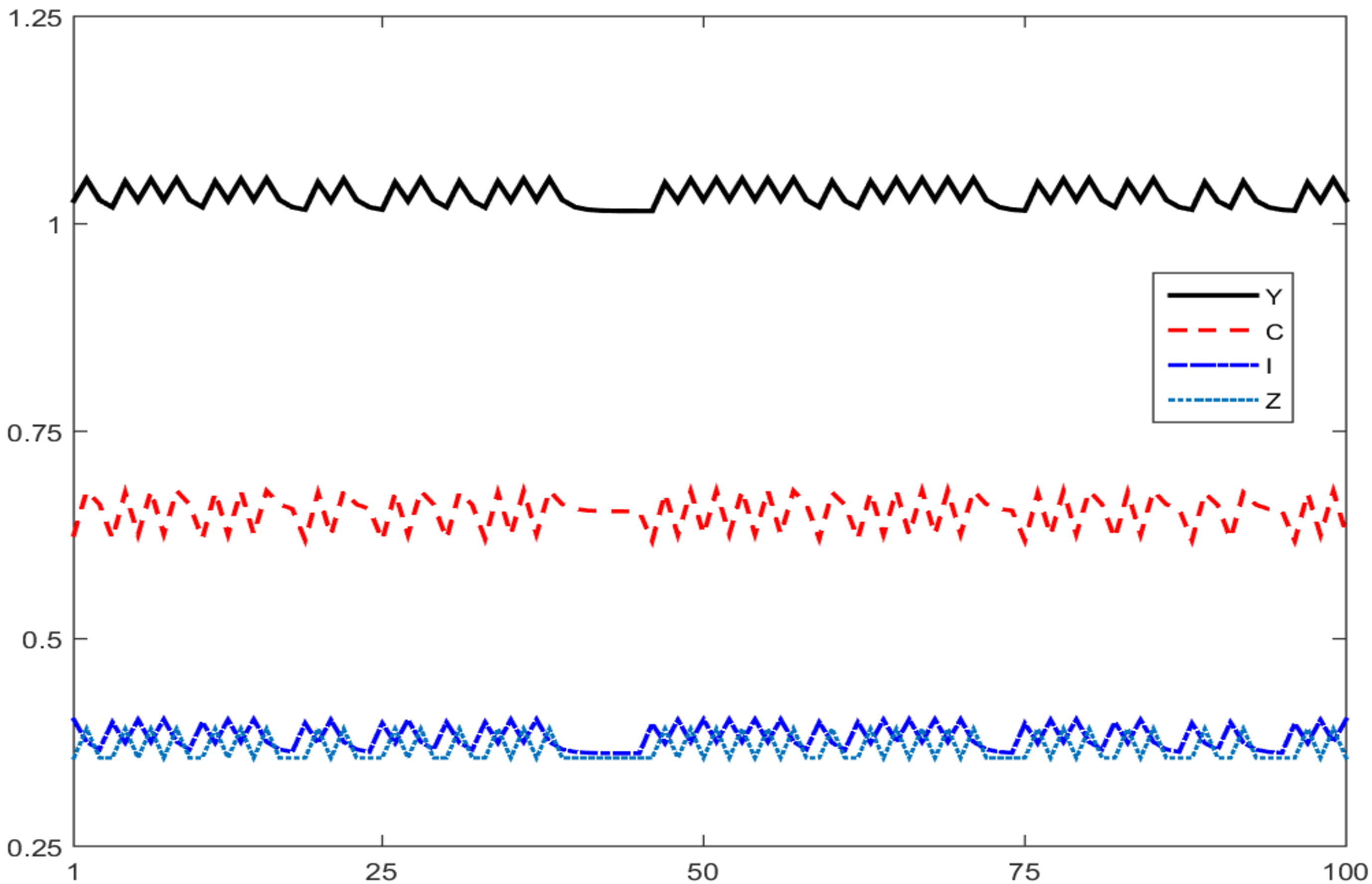




Simulated series with constant probability: π_t=0.5

Simulated output (Y), consumption (C) and investment (I) normalized by steady state output

Excessive volatility can be reduced by assuming that the probability of an investment bust next period rises once the current investment output Z_t crosses a threshold.

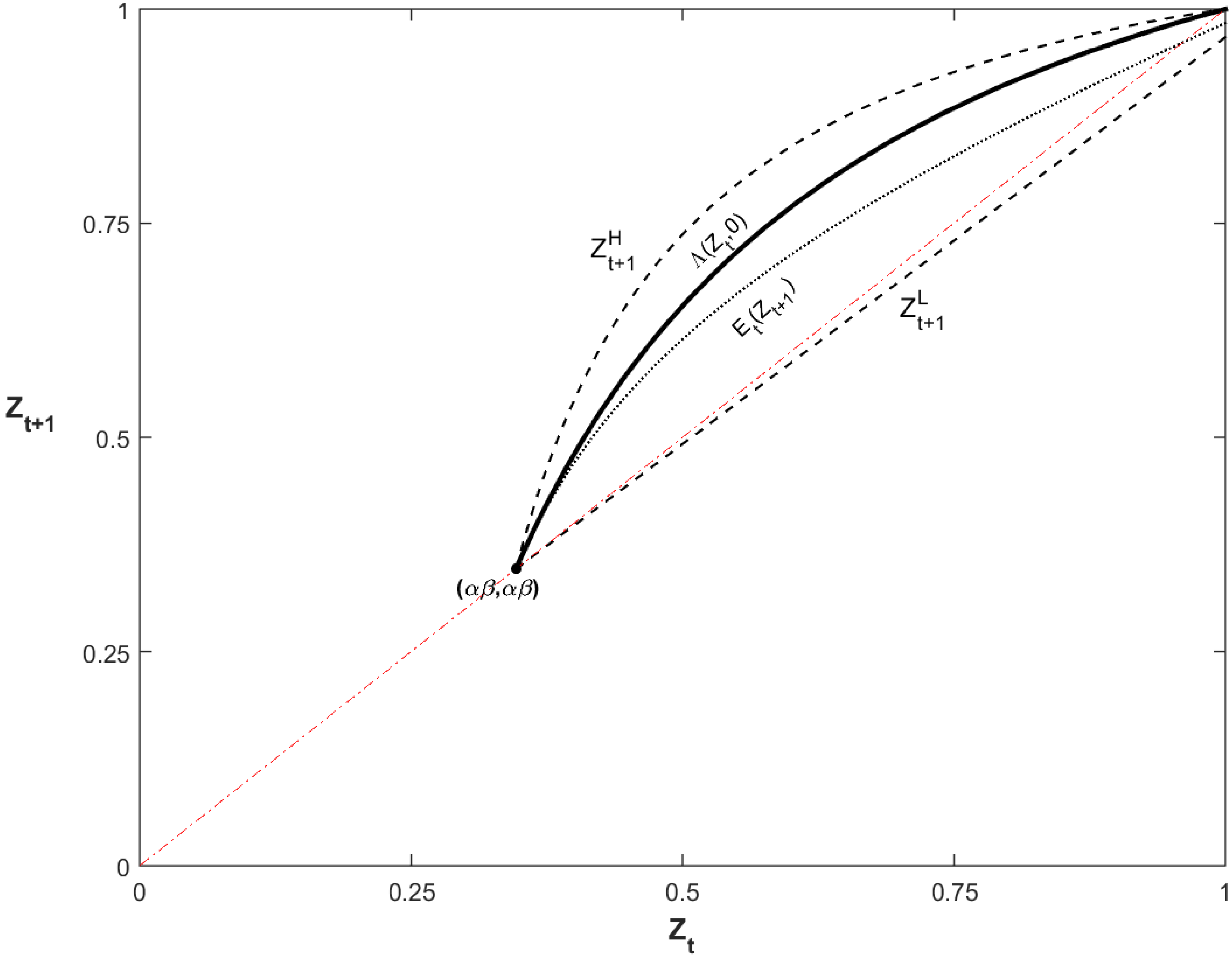


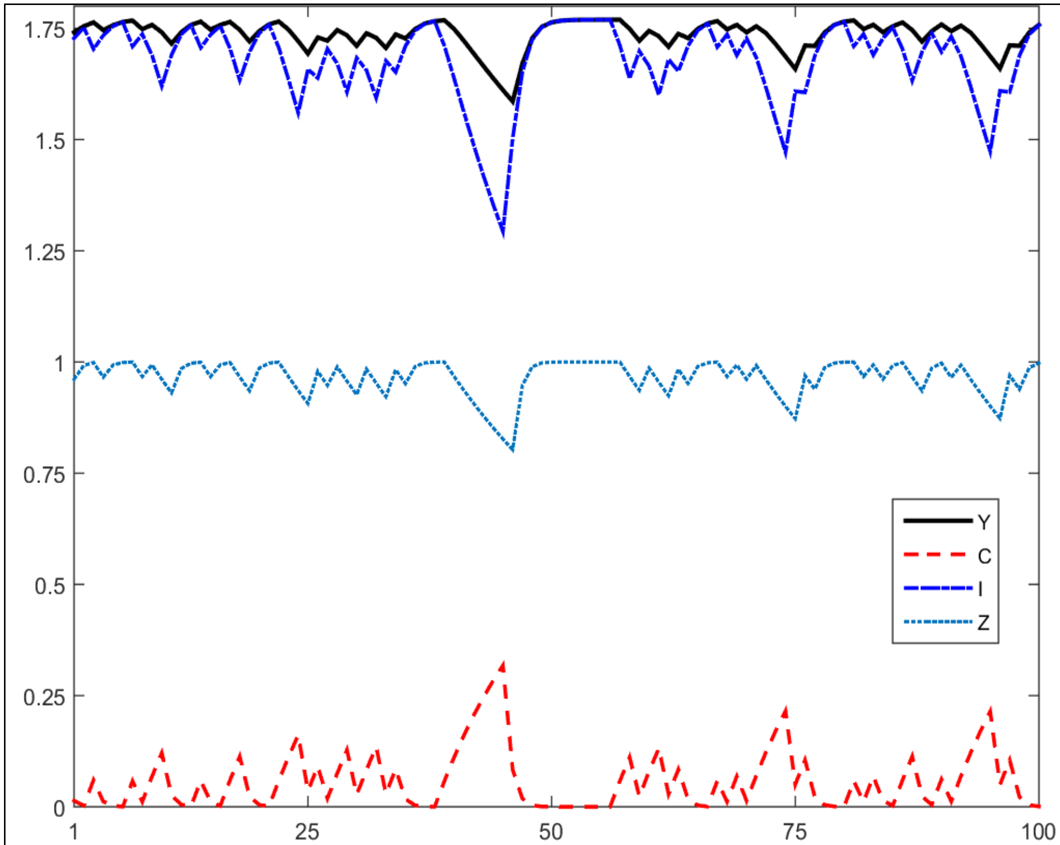
Simulated series with state-contingent probability of bust:

$\pi_t = 0.5$ for $\alpha\beta + \Delta = 0.356 \leq Z_t \leq 0.36$ & $\pi_t = 1 - 10^{-100}$ for $Z_t > 0.36$

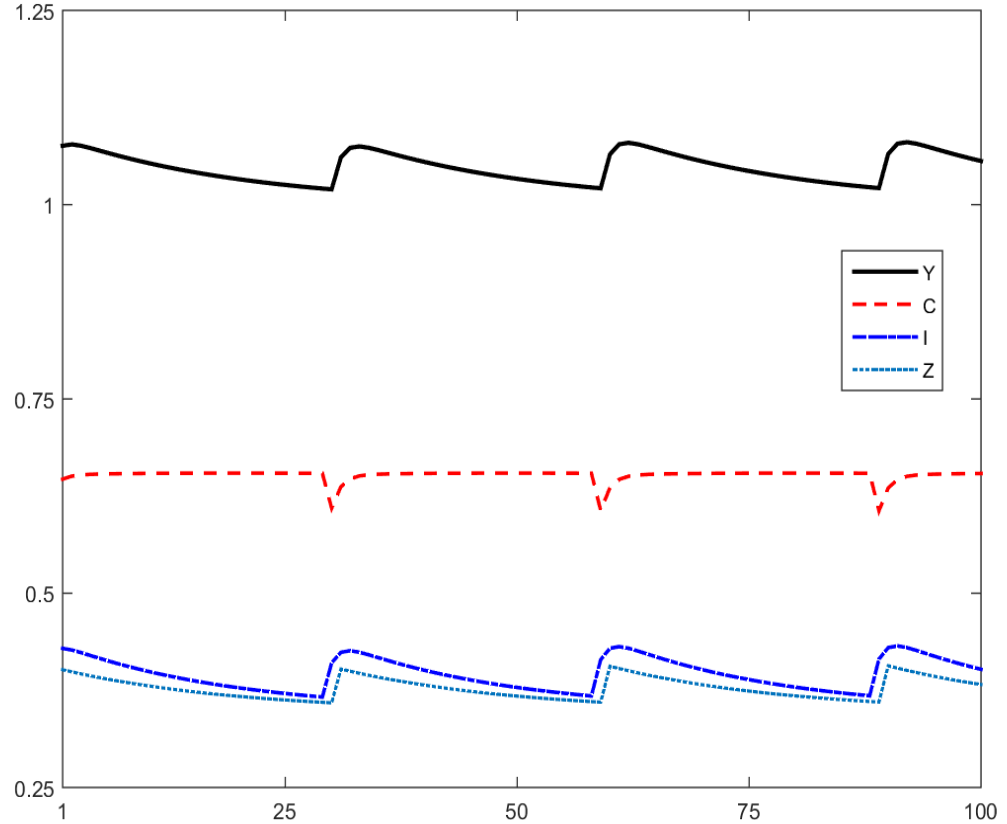
Specification II: gradual contractions

$$Z_{t+1}^{L}=\alpha\beta+0.95\cdot(Z_t-\alpha\beta),\ \pi=0.5$$





Simulated series with constant probability of bust: $\pi_t=0.5$



Simulated series with state-contingent probability of bust: $\pi_t=0.5$ **for** $\alpha\beta+\Delta=0.356\leq Z_t\leq 0.36$ **&** $\pi_t\simeq 1$ **for** $Z_t>0.36$

Table 1. Long-Plosser model with bubbles: predicted business cycle statistics

	Standard dev. %			Corr. with Y		Autocorr.			Mean (% deviation from SS)			
	Y	C	I	C	I	Y	C	I	Y	C	I	Z
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
(a) Specification I: $Z_t^L = \alpha\beta + \Delta$												
$\pi_t=0.5$	11.72	100.19	33.48	-0.42	0.62	0.62	0.47	0.62	13.49	-7.62	53.31	31.15
$\pi_t \cong 1$ for $z_t>0.36$	1.33	3.51	3.82	0.77	-0.26	-0.26	-0.66	-0.26	3.27	-0.13	9.71	6.25
(b) Specification II: $Z_t^L = \alpha\beta + 0.95\times(z_t - \alpha\beta)$												
$\pi_t=0.5$	1.73	210.28	4.94	-0.31	0.68	0.68	0.46	0.68	73.09	-89.68	380.11	177.21
$\pi_t \cong 1$ for $z_t>0.36$	1.40	1.30	4.00	0.14	0.85	0.85	0.28	0.85	4.45	-0.26	13.34	8.46
(c) US Data (from King and Rebelo (1999))												
	1.81	1.35	5.30	0.88	0.80	0.88	0.80	0.87				

Example II: RBC model with incomplete capital depreciation & endogenous labor

$Max\ E_0\sum_{t=0}^{\infty}\beta^s u(C_t,L_t)$ subject to resource constraint

$C_t+I_t=Y_t$ with $I_t=K_{t+1}-(1-\delta)K_t$ and $Y_t=\theta(K_t)^{\alpha}(L_t)^{1-\alpha}$.

$u(C_t,L_t)=(1-\sigma)^{-1}(C_t-\nu(L_t))^{1-\sigma},\nu(L_t)=(\Psi/(1+1/\eta))\{L_t^{1+1/\eta}-L^{1+1/\eta}\},$

$(1-\alpha)Y_t/L_t=\Psi\cdot(L_t)^{1/\eta}\Rightarrow L_t=n(K_t,\theta_t)\equiv(\theta_t(K_t^{\alpha})(1-\alpha)/\Psi)^{1/(\alpha+1/\eta)}$

$E_t\beta\{(C_{t+1}-\nu(L_{t+1}))^{-\sigma}/(C_t-\nu(L_t))^{-\sigma}\}(\alpha Y_{t+1}/K_{t+1}+1-\delta)=1$

$\beta\{(C_{t+1}-\nu(L_{t+1}))^{-\sigma}/(C_t-\nu(L_t))^{-\sigma}\}(\alpha Y_{t+1}/K_{t+1}+1-\delta)=1+\varepsilon_{t+1},$

ε_{t+1}**: sunspot with** $E_t\varepsilon_{t+1}=0$

$\Rightarrow\ K_{t+2}=\kappa(K_{t+1},K_t,\varepsilon_{t+1})$

$$K_{t+2}=\kappa(K_{t+1},K_t,\varepsilon_{t+1})$$

Assume ε_{t+1} takes only two values, $-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\pi/(1-\pi)$, with probabilities π & $1-\pi$; $\overline{\varepsilon_t}\in[0,1]$.

'Bust': $K^L_{t+2}=\lambda(K_{t+1})$ **no-sunspot decision rule**

$$K^L_{t+2}=\kappa(K_{t+1},K_t,-\overline{\varepsilon_t})$$

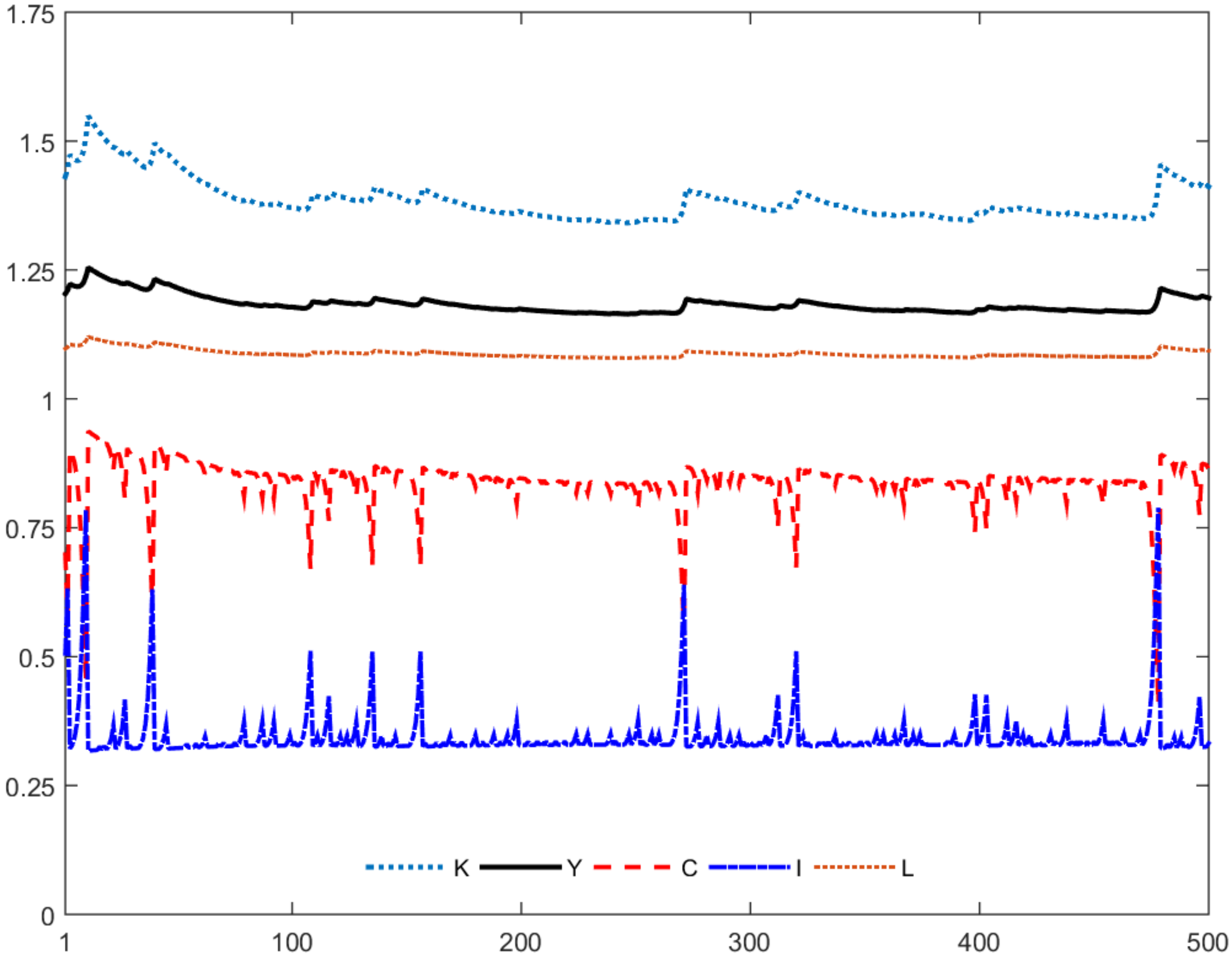
'Boom': $K_{t+2}=\kappa(K_{t+1},K_t,\overline{\varepsilon_t}\pi/(1-\pi))$

Parameters:

$\beta=0.99$, $\alpha=0.35$, $\delta=0.025$ (depreciation rate)

$\sigma=\eta=1$ (risk aversion, labor supply elasticity

$\pi=0.5$ (bust probability)



Non-linear RBC model (incomplete capital depreciation, variable labor) with bubbles. Simulated paths of GDP (Y), consumption (C), investment (I) are normalized by steady state GDP. Capital (K) and hours series (L) are normalized by their respective steady states.

Table 2. RBC model (incomplete capital depreciation) with bubbles: predicted business cycle statistics

	Standard dev. %				Corr. with Y			Autocorr.				Mean (% deviation from SS)				
	Y	*C*	*I*	*L*	*C*	*I*	*L*	*Y*	*C*	*I*	*L*	*Y*	*C*	*I*	*L*	*K*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*	*(16)*
π_t=0.5	0.46	9.67	11.12	0.23	0.13	-0.21	1.00	0.91	0.54	0.52	0.91	18.6	11.8	39.0	8.90	39.0

Example III: Small Open Economy

$Max\ E_0 \sum_{t=0}^{\infty} \beta^s \ln(C_t)$ **s.t.** $C_t + A_{t+1} = A_t \cdot R_t + Y$

C_t: consumption; Y: output

A_{t+1}: net foreign assets (NFA); R_t: gross interest rate;

- Euler equation: $\beta E_t (C_{t+1}/C_t)^{-1} R_{t+1} = 1$
- $R_t = R(A_t)$, with $R' < 0$

■ No-sunspots solution: policy function $A_{t+1} = \lambda(A_t)$

■ How to construct sunspot equilibrium:

Euler equation $\Rightarrow \beta(C_{t+1}/C_t)^{-1}R(A_{t+1})=1+\varepsilon_{t+1};$

ε_{t+1}: sunspot with $E_t\varepsilon_{t+1}=0.$

$\Rightarrow C_{t+1}=C_t\cdot\beta\cdot R(A_{t+1})/(1+\varepsilon_{t+1})$

Budget constraint: $C_t=A_t\cdot R(A_t)+Y-A_{t+1}$

$\Rightarrow A_{t+2}=A_{t+1}R(A_{t+1})+Y-(A_tR(A_t)+Y-A_{t+1})\cdot\{\beta R(A_t)/(1+\varepsilon_{t+1})\}^{1/\sigma}$

$A_{t+2}=\kappa(A_{t+1},A_t,\varepsilon_{t+1})$: NFA law of motion with sunspot

$\kappa_\varepsilon>0,\ \kappa_{\varepsilon\varepsilon}>0$: non-linearity permits sunspot equil.

- $A_{t+2}=\lambda(A_{t+1})$: NFA law of motion without sunspot

- Gap between NFA with & without sunspot:

$g_{t+1}\equiv A_{t+2}-\lambda(A_{t+1})$. In linearized system: gap explodes

$g_{t+1}=(\kappa_1-\lambda')\cdot g_t+\kappa_3\cdot\varepsilon_{t+1};\ (\kappa_1-\lambda')>1$

In non-linear system: gap can be stationary
Non-sunspot solution as 'attractor'

$A_{t+2}=\kappa(A_{t+1},A_t,\varepsilon_{t+1})$: NFA law of motion with sunspot

Assume ε_{t+1} takes only two values, $-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\pi_t/(1-\pi_t)$, with probabilities π_t & $1-\pi_t$; $\overline{\varepsilon_t}\in[0,1]$.

'Bust': $A^L_{t+2}=\lambda(A_{t+1})+\Delta$. **No-sunspot decision rule.** $\Delta>0$

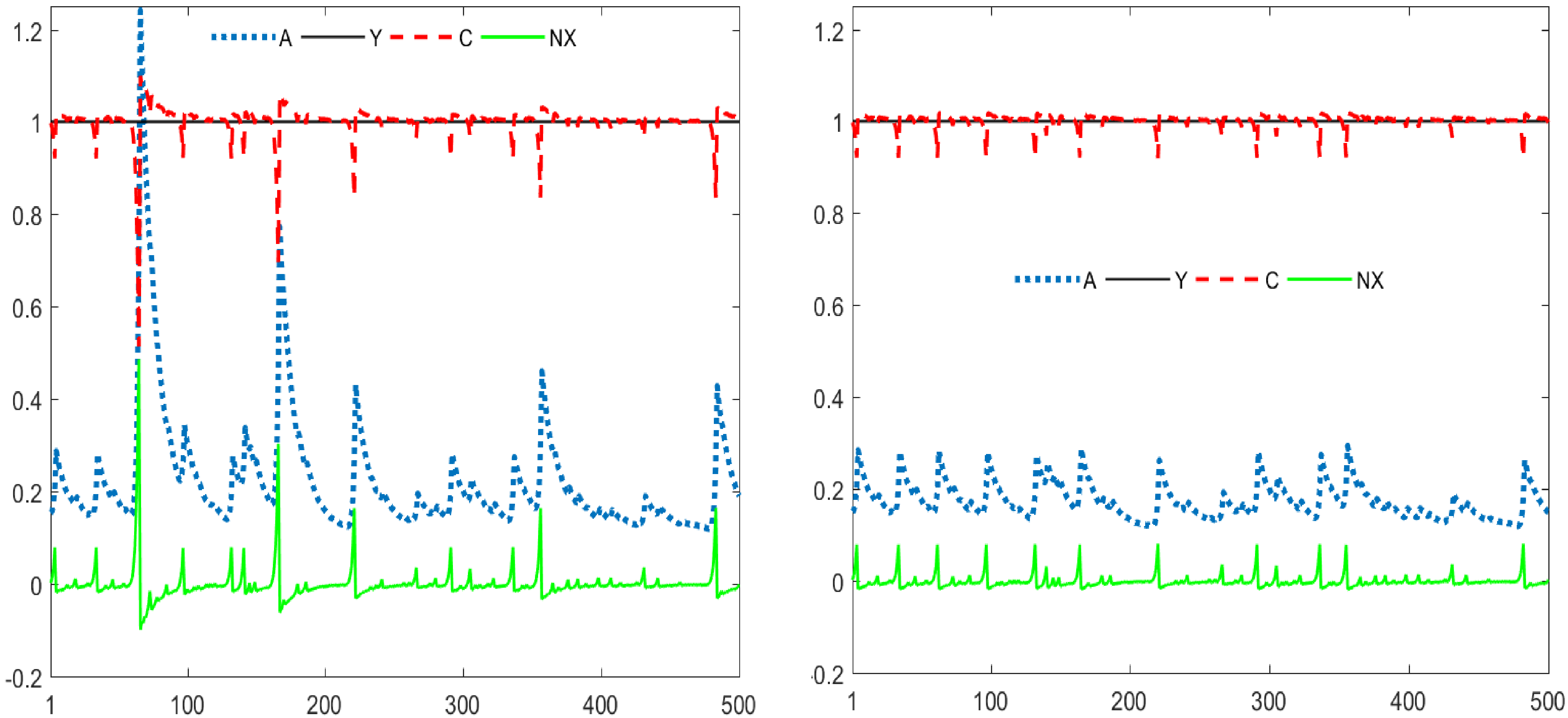
$A^L_{t+2}=\kappa(A_{t+1},A_t,-\overline{\varepsilon_t})$: this pins down $\overline{\varepsilon_t}$

'Boom': $A^H_{t+2}=\kappa(A_{t+1},A_t,\overline{\varepsilon_t}\pi/(1-\pi))$

Parameters:

$\beta=0.99;\ R(A_t)=\exp(-a\cdot A_t/Y)/\beta,\quad a=0.01.$

$\pi_t=0.5$ (bust probability)



(a) Constant bust probability: $\pi_t = 0.5$

(b) State-contingent bust probability:

$\pi_t = 0.5$ **for** $A_{t+1}/Y \leq 0.25$;

$\pi_t \simeq 1$ **for** $A_{t+1}/Y > 0.25$

Non-linear Small Open Economy model with bubbles

Simulated paths of net foreign assets (A), GDP (Y), consumption (C) and net exports (NX). All series normalized by steady state GDP.

Table 3. Small Open Economy model (endowments) with bubbles: predicted business cycle statistics

	Standard dev. %			Corr. with Y		Autocorr.			Mean (% deviation from SS)			
	A	C	NX	C	NX	A	C	NX	A	Y	C	NX
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
π_t=0.5	11.43	9.07	4.95	---	---	0.90	0.53	0.52	24.44	0.00	0.09	-0.09
$_t\cong 1$ for $A_{t+1}/y>0.25$	2.07	1.30	1.27	---	---	0.81	0.26	0.26	16.70	0.00	0.13	-0.13

CONCLUSIONS

- Stationary sunspot equilibria exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions.
- In the sunspot equilibria considered here, the economy temporarily diverges from the no-sunspots trajectory, before abruptly reverting towards that trajectory.
- In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity.

ADDITIONAL MATERIAL

Blanchard (1979):

$$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1 \quad \Rightarrow \quad y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

How non-linearity may generate stationary bubble:

Assume: $E_t \exp(z_{t+1} - \lambda z_t) = a, \quad \lambda > 1, \; a > 0$

$\Rightarrow \exp(z_{t+1} - \lambda z_t) = a + \eta_{t+1}$ with $E_t \eta_{t+1} = 0$

$\Rightarrow z_{t+1} = \lambda z_t + \log(a + \eta_{t+1})$. Let $y_t \equiv z_t + \ln(a)/(\lambda - 1), \; \varepsilon_{t+1} \equiv \eta_{t+1}/a$

$$\Rightarrow y_{t+1} = \lambda \cdot y_t + \ln(1 + \varepsilon_{t+1}), \quad E_t \varepsilon_{t+1} = 0$$

y_{t+1} is **concave** in $\varepsilon_{t+1} \Rightarrow E_t y_{t+1} < \lambda \cdot y_t$

Let $\varepsilon_{t+1} \in \{-\overline{\varepsilon}_t; \; \overline{\varepsilon}_t \pi/(1-\pi)\}$ with prob. $\pi, 1-\pi$. $\overline{\varepsilon}_t > 0$

Set $\overline{\varepsilon}_t \in [0,1)$ so that $y_{t+1} = \lambda \cdot y_t + \ln(1 - \overline{\varepsilon}_t) = \Delta$

$y_{t+1} = y^H_{t+1} \equiv \lambda \cdot y_t + \ln\{1 + [1 - \exp(\Delta - \lambda \cdot y_t)] \cdot \pi/(1-\pi)\}$ with prob. $1-\pi$

$y_{t+1} = \Delta$ with probability π