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Rational Bubbles in Non-Linear Business Cycle Models

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This paper shows that stationary sunspot equilibria exist in completely standard *non-linear* DSGE models, even when the *linearized* versions of those models have a unique solution. Thus, those models have additional stationary solutions, if non-linearity is considered. The Blanchard and Kahn (1980) conditions are, hence, irrelevant for non-linear models. In the sunspot equilibria considered here, the economy may temporarily diverge from the no-sunspots trajectory, before abruptly reverting towards that trajectory. In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity. Numerical simulations suggest that simple non-linear DSGE models driven by stationary bubbles can generate persistent fluctuations of real activity and capture key business cycle stylized facts.

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1. Introduction

Linearized dynamic stochastic general equilibrium (DSGE) models with a unique stable solution are the workhorses of modern quantitative macroeconomics (e.g., Kollmann et al. (2011a,b)). This paper shows that stationary sunspot equilibria may exist in standard non-linear DSGE models, even when the linearized versions of those models have a unique solution. To explain the basic intuition for this result, consider the simple case of a model with just one variable, namely a jump variable whose period t value is denoted Y_t. Assume that Y_t obeys the expectational non-linear difference equation

$$E_t G(Y_{t+1}, Y_t) = 0, \tag{1}$$

where E_t denotes the mathematical expectation, conditional on period t information, and $G\colon R^2 \to R$ is a function. [1] Linearization of (1) around a deterministic steady state (i.e. around a value Y that satisfies $G(Y,Y)=0$) gives:

$$E_t y_{t+1} = \lambda \cdot y_t, \tag{2}$$

with $y_t \equiv Y_t - Y$ and $\lambda \equiv -G_2/G_1$, where G_1 and G_2 are derivatives of the function G with respect to its first and second arguments, respectively (these derivatives are evaluated at the steady state). [2] The linearized model has a unique non-explosive solution if and only if $|\lambda| > 1$; that unique solution is: $y_t = 0$ (see Blanchard and Kahn (1980), Proposition 1).

This paper shows that, even when $|\lambda| > 1$ holds, the *non-linear* model (1) can have multiple stationary solutions. Note that (1) holds whenever $G(Y_{t+1}, Y_t) = \varepsilon_{t+1}$, where ε_{t+1} is a random variable with $E_t \varepsilon_{t+1} = 0$. Express Y_{t+1} as a function of Y_t and ε_{t+1}:

$$Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1}). \tag{3}$$

As this simple model has not exogenous forcing variables, the disturbance ε_{t+1} solely reflects unanticipated changes in Y_{t+1} that are driven by changes in agents' expectations about the future path $\{Y_{t+s}\}_{s>1}$. I refer to ε_{t+1} as a sunspot.

If an arbitrary white noise process $\{\varepsilon_{t+1}\}$ is fed into (3), then $\{Y_{t+1}\}$ diverges when $|\Lambda_Y| > 1$. The key contribution of the paper is to show how to construct a process $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1} = 0$ such that the process $\{Y_{t+1}\}$ generated by (3) is stationary, even when $|\Lambda_Y| > 1$. There are two requirements for the existence of a stationary solution: (i) the recursion (3) has to be non-linear in ε_{t+1} ($\Lambda_{\varepsilon\varepsilon} \neq 0$); (ii) The distribution of ε_{t+1} has to depend on Y_t. [3]

To see this intuitively, take a second-order expansion of (3) w.r.t. ε_{t+1}: $Y_{t+1} \cong \Lambda(Y_t, 0) + \Lambda_\varepsilon(Y_t, 0) \cdot \varepsilon_{t+1} + \tfrac{1}{2} \Lambda_{\varepsilon\varepsilon}(Y_t, 0) \cdot (\varepsilon_{t+1})^2$, which implies $E_t Y_{t+1} \cong \Lambda(Y_t, 0) + \tfrac{1}{2} \Lambda_{\varepsilon\varepsilon}(Y_t, 0) \cdot E_t(\varepsilon_{t+1})^2$.

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[1] An example for (1) is the Cagan model in which the demand for real balances at date t depends on the expected inflation rate, so that the price level at t depends on the distribution of the price level at t+1 (e.g., Obstfeld and Rogoff (1996), ch.8.2.).

[2] Throughout, a function with subscripts denotes its partial derivatives.

[3] Bacchetta et al. (2012) study a stylized asset pricing model with two-period lived agents in which stationary stock price bubbles can arise if the sunspot shock is heteroscedastic. The work here highlights the importance of heteroskedastic sunspot shocks, for generating multiple equilibria, in a DSGE business cycle models.

Assume that the variance of ε_{t+1} is a function of Y_t: $E_t(\varepsilon_{t+1})^2 = f(Y_t) \geq 0$. Then $E_t Y_{t+1} \cong M(Y_t)$, with $M(Y_t) \equiv \Lambda(Y_t, 0) + \frac{1}{2}\Lambda_{\varepsilon\varepsilon}(Y_t, 0) \cdot f(Y_t)$. Provided $\Lambda_{\varepsilon\varepsilon} \neq 0$ there may exist a function $f(Y_t)$ such that $|M'| < 1$ holds, even when $|\Lambda_Y| > 1$. Thus $\{Y_{t+1}\}$ may exhibit mean reversion, even when $|\Lambda_Y| > 1$. The non-linearity (in ε_{t+1}) of the function Λ implies that the conditional distribution of Y_{t+1} is asymmetric. For example, assume that $\Lambda_Y(Y_t, 0) > 1$ and that $\Lambda(Y_t, \varepsilon_{t+1})$ is strictly concave in ε_{t+1} $(\Lambda_{\varepsilon\varepsilon} < 0)$. Then the variance of the sunspot ε_{t+1} has to be an increasing function of Y_t, in order to ensure mean reversion; also, for a given Y_t, the conditional distribution of Y_{t+1} will be left-skewed: the absolute value of (unexpected) contractions in Y_{t+1} will exceed the size of expansions in Y_{t+1}.

The remainder of this paper uses these ideas to construct stationary sunspot equilibria for standard fully-fledged non-linear DSGE business cycle models.

The multiple equilibria identified here have similarities and important differences, compared to the bubbles in linearized models analyzed by Blanchard (1979) and Blanchard and Watson (1982). Like the Blanchard bubbles, the endogenous variables can temporarily diverge from the no-sunspot trajectories, before abruptly reverting towards those trajectories. A key difference is that the bubbles in non-linear models considered here are stationary, while Blanchard's bubbles exhibit explosive *expected* trajectories $(\lim_{\tau\to\infty} E_t y_{t+\tau} = \pm\infty)$.

A sizable literature that developed macro DSGE models for which $|\lambda| \leq 1$ holds, so that sunspot equilibria exist (Blanchard and Kahn (1980), Prop. 3). When $|\lambda| \leq 1$ holds, then $y_{t+1} = \lambda \cdot y_t + \eta_{t+1}$ is a stable solution of (2) for any stationary random variable η_{t+1} with $E_t \eta_{t+1} = 0$. Trivially, non-linear versions of models with $|\lambda| \leq 1$ too have sunspot equilibria. Linearized dynamic macro models may exhibit stationary sunspot equilibria if increasing returns, externalities (see, e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999)), financial frictions (e.g., Martin and Ventura (2018)) and/or overlapping generations population structures (e.g., Galí (2018)) are assumed. The specific assumptions and calibrations that deliver these stationary sunspot equilibria can be debatable. By contrast, the paper here argues that very standard DSGE model (without the features that were just mentioned) can deliver stationary sunspot equilibria, if non-linear effects are considered.

I show that canonical non-linear DSGE models can have sunspot equilibria, **even** when the linearized model version has a unique solution. This is important because linearized DSGE models with unique solutions are the workhorses of modern macroeconomics. The large literature on estimated DSGE models has (almost) entirely focused on linearized models with unique solutions (e.g., Kollmann et al. (2016)). The results here suggest that those models may have additional stationary solutions, if non-linearity is taken into consideration.

The model equation (1) reflects agents' optimal intertemporal choices between periods t and t+1. In models with infinitely lived optimizing agents there is an addition set of optimality conditions besides (1), namely transversality conditions (TVCs) that pertain to the trajectory of state variables at an infinite horizon. For example, in an economy with capital accumulation, TVCs requires that the value of the capital stock is zero, at infinity--otherwise there is capital over-accumulation, and the economy is 'dynamically inefficient' (e.g., Abel et al. (1989)). Explosive bubbles of the Blanchard (1979) type violate TVCs. I abstract from TVCs in this paper. The goal is to identify stationary sunspot equilibria for non-linear dynamic models. The canonical DGSE models discussed below are usually presented as assuming infinitely lived agents. I show that there are overlapping generations (OLG) models that deliver the same set of intertemporal equilibrium conditions. However, optimizing household behavior in those models does not obey a TVC. In an economy with stationary bubbles, detecting violations of TVCs can be challenging, as over-accumulation may occur only in certain states of the world, possibly with infinitesimal probability. Agents may not have the computing power to detect this.

2. Examples

2.1. Long-Plosser RBC model

This Section considers a one-sector version of the Long and Plosser (1983) model, i.e. a simple RBC model with log utility and full capital depreciation per period. I provide a detailed discussion of that model, as it allows closed form solutions with bubbles. The period utility function and the production function are $u(C_t)=\ln(C_t)$ and $Y_t=\theta_t K_t^{\alpha}$, $0<\alpha<1$, where $C_t, Y_t, K_t>0$ and $\theta_t>0$ are period t consumption, output, capital and exogenous total factor productivity (TFP). The labor supply is assumed fixed (normalized at unity).[4] The resource constraint is $C_t+I_t=Y_t$, where I_t is gross investment. As the capital depreciation rate is 100%, gross investment equals next period's capital stock: $I_t=K_{t+1}$. The household's Euler equation is

$$E_t\beta(C_t/C_{t+1})\alpha Y_{t+1}/K_{t+1}=1, \tag{4}$$

where $0<\beta<1$ is the household's subjective discount factor. Substituting the resource constraint into the Euler equation, we obtain an expectational difference equation in the investment/output ratio $Z_t \equiv K_{t+1}/Y_t$:

$$\alpha\beta E_t[(1-Z_t)/(1-Z_{t+1})]/Z_t = 1. \tag{5}$$

A constant investment share $Z_t=Z_{t+1}=\alpha\beta$ solves (5). Under this solution, consumption and investment are constant shares of output: $C_t=(1-\alpha\beta)Y_t$, $K_{t+1}=(1-\alpha\beta)Y_t$. This corresponds to the textbook solution of the Long-Plosser model (see, e.g., Blanchard and Fischer (1989)).

Linearization of (5) around $Z=\alpha\beta$ gives: $E_t z_{t+1}=\lambda z_t$, where $z_t \equiv Z_t - Z$ and $\lambda=1/(\alpha\beta)>1$. Thus, $Z_t=\alpha\beta\ \forall t$ is the unique non-explosive solution of the **linearized model.**

I show now that the **non-linear model** has other stationary solutions. Note that (5) holds for any process $\{Z_t\}$ such that $\alpha\beta[(1-Z_t)/(1-Z_{t+1})]/Z_t = 1+\varepsilon_{t+1}$, where ε_{t+1} is a random variable with zero conditional mean: $E_t\varepsilon_{t+1}=0$. This can be written as:

$$Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1}) \equiv 1-\alpha\beta(1/Z_t-1)/(1+\varepsilon_{t+1}). \tag{6}$$

Consumption and investment are positive when $0<Z_t<1\quad \forall t$. Z_{t+1} is an increasing, concave function of Z_t and of ε_{t+1}. Fig.1 plots Z_{t+1} as a function of Z_t, and that for three values of ε_{t+1}: $\varepsilon_{t+1}=0$ (thick black line), $\varepsilon_{t+1}=0.5$ and $\varepsilon_{t+1}=-0.5$ (thin dashed lines). Throughout this paper, I set α=0.35 and β=0.99, so that $\alpha\beta=0.3465$; these parameter values are standard in quarterly business cycle models.

[4] With endogenous hours, and a period utility function that is additively separable in consumption and hours, the Long-Plosser model without bubbles predicts that hours are constant. Under endogenous hours, the Long-Plosser model generates fluctuations in hours.

Fig. 1 shows that the function $Z_{t+1}=\Lambda(Z_t,0)$ cuts the 45-degree line from below, at $Z_t=Z_{t+1}=\alpha\beta$. When $Z_t<\alpha\beta$, then the model can hit a zero-capital corner solution in subsequent periods. I thus restrict attention to solutions for which $\{Z_\tau\}$ stays forever in the interval $[\alpha\beta,1)$. This requires that the support of the distribution of ε_{t+1} has to be bounded below: $\varepsilon_{t+1}\geq -1+[\alpha\beta/(1-\alpha\beta)]\cdot[1/Z_t-1]$. Thus, the distribution of ε_{t+1} has to depend on Z_t.

For simplicity, assume that ε_{t+1} only takes the two values $-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\cdot\pi_t/(1-\pi_t)$ with probabilities π_t and $1-\pi_t$, respectively, where $\overline{\varepsilon_t}\in[0,1]$. Z_{t+1} then takes these two values:

$$Z_{t+1}^L\equiv\Lambda(Z_t,-\overline{\varepsilon_t}) \text{ and } Z_{t+1}^H\equiv\Lambda(Z_t,\overline{\varepsilon_t}\pi_t/(1-\pi_t)) \text{ with } Z_{t+1}^L\leq Z_{t+1}^H\leq 1. \quad (7)$$

The first of these equations can be solved for $\overline{\varepsilon_t}$, as a function of Z_{t+1}^L and Z_t:

$$\overline{\varepsilon_t}=\Lambda(Z_t,-Z_{t+1}^L)\,. \quad (8)$$ [5]

In what follow, I postulate that $Z_{t+1}^L=f(Z_t)$ is a non-decreasing function of Z_t satisfying the following restriction:

$$\alpha\beta\leq f(Z_t)\leq\Lambda(Z_t,0) \qquad \text{for } Z_t\in[\alpha\beta,1)\,. \quad (9)$$

Substitution of $Z_{t+1}^L=f(Z_t)$ into (10) and (9) then pins down $\overline{\varepsilon_t}$ and Z_{t+1}^H:

$$\overline{\varepsilon_t}=\overline{\varepsilon}(Z_t)\equiv\Lambda(Z_t,-f(Z_t)) \text{ and } Z_{t+1}^H=Z^H(Z_t,\pi_t)\equiv\Lambda(Z_t,\overline{\varepsilon_t}(Z_t)\pi_t/(1-\pi_t)). \quad (10)$$

When $Z_t\in[\alpha\beta,1)$, then restriction (9) guarantees that $Z_\tau\in[\alpha\beta,1)$ holds $\forall\,\tau>t$.

In what follows, I discuss two illustrative specifications of $Z_{t+1}^L=f(Z_t)$ that meet restriction (9).

2.1.1. Specification I: sharp contractions

Specification I postulates that $Z_{t+1}^L=\alpha\beta+\Delta$, where $\Delta\in(0,1-\alpha\beta)$ is a constant. A strictly positive value of Δ is needed to allow uninterrupted random fluctuations. For assume that $\Delta=0$ and consider what happens if $Z_t=\alpha\beta$. Then $Z_{t+1}^H=Z_{t+1}^L=\alpha\beta$, i.e. Z is forever stuck at $\alpha\beta$. Setting $\Delta>0$ rules out that absorbing state.

The numerical simulations presented below set Δ at 1% of steady state output, Y: $\Delta=0.01\times Y$. The baseline variant of model specification I assumes a constant probability: $\pi_t=0.5\ \forall t$. Panel (a) of Fig. 2 plots Z_{t+1}^L, Z_{t+1}^H and $E_tZ_{t+1}=\pi_tZ_{t+1}^L+(1-\pi_t)Z_{t+1}^H$, as functions of Z_t. Also shown in Panel (a) is the value of Z_{t+1} that would obtain without random sunspot $(\varepsilon_{t+1}=0)$: $Z_{t+1}=\Lambda(Z_t,0)$. The investment/output ratio grows between t and t+1 $(Z_{t+1}>Z_t)$ when $\varepsilon_{t+1}=\overline{\varepsilon_t}\cdot\pi_t/(1-\pi_t)>0$; when $\varepsilon_{t+1}=-\overline{\varepsilon_t}$, the investment rate either remains unchanged at $\alpha\beta+\Delta$ (if $Z_t=\alpha\beta+\Delta$), or it drops to $Z_{t+1}=\alpha\beta+\Delta$ (if $Z_t>\alpha\beta+\Delta$).

[5] Note that $Z_{t+1}^L=1-\alpha\beta(1/Z_t-1)/(1-\overline{\varepsilon_t})$. Thus, $\overline{\varepsilon_t}=1-\alpha\beta(1/Z_t-1)/(1-Z_{t+1}^L)$. Using the function Λ defined in (7), we can write this as $\overline{\varepsilon_t}=\Lambda(Z_t,-Z_{t+1}^L)$.

The model can thus generate runs of continued increases in the investment ratio, that are followed by abrupt contractions in the investment ratio. Fig. 2 shows that Z_{t+1}^{H} is a steeply increasing function of Z_t. Thus, as long as the bubble continues to grow, it triggers a rapid rise in the investment/output ratio. If $Z_t = Z_t^L$, then an uninterrupted 'bubble run' in periods $\tau > t$ (triggered by a sequence of positive draws ε_τ) induces these successive values of Z: 0.3916, 0.5373, 0.8058, 0.9554, 0.9918, 0.9985, … .

Panel (b) of Fig. 2 shows simulated sample paths of output, consumption, gross investment (I) and of the investment/output ratio (Z). In order to assess whether the bubble alone can generate a realistic business cycle, I assume that TFP is constant. The Figure shows that the model generates massive swings in investment and output. During an expanding bubble, the rapid rise in investment is accompanied by a contraction in consumption.

Table 1 reports moments of HP filtered logged time series generated by the model. Line 1 of Panel (a) shows moments for specification I, with probability $\pi_t = 0.5$. Predicted moments are based on a simulation run of 10^6 periods. The predicted standard deviation of output is 11.7% which is about five times larger than the historical standard deviation of quarterly GDP in advanced economies (e.g., Backus et al. (1992)). The model-predicted volatility of consumption and investment too is excessive, when compared to the data. The model predicts that output, consumption and investment are serially correlated. However, consumption is predicted to be countercyclical, which is inconsistent with the data.

The baseline model generates a mean capital stock (=investment) that is 53.3% higher than the capital stock in the deterministic steady state, while mean output is 13.5% higher and mean consumption is 7.6% lower than in the deterministic steady state (see Cols. (9)-(12) in Table 1).

Recall that the model variant above assumes a constant 50% probability that the bubble grows next period. The model predicts smaller, more realistic, fluctuations in real activity occur if we assume that the probability of growth in the bubble falls once the investment/output ratio exceeds a threshold. As an illustration, assume that π_t is very close to unity, for values of the investment/output ratio greater than 0.36.[6] This threshold is chosen as it generates (more) realistic output volatility. It implies that the investment/output ratio oscillates between these two values: 0.3565 and 0.3916 (see below). Note that the 'High' investment ratio exceeds the 'Low' ratio by about 10%. When the investment/output ratio at date *t* takes the 'Low' value $Z^L \equiv \alpha\beta + \Delta = 0.3565$, then next period's investment ratio is either 'Low' (Z^L) of 'High' ($Z^H(\alpha\beta + \Delta, 0.5) = 0.3916$) with 50% probability. If the date *t* investment ratio is 'High', then the investment ratio falls to the 'Low' value in the next period with probability arbitrarily close to 1. Panel (c) of Figure 2 shows simulated sample paths generated for this model version, and the second Line in Panel (a) of Table 1 reports the corresponding model-predicted business cycle statistics. This model variant produces output fluctuations that are more in line with the data (predicted standard deviation of GDP: 1.33%), however now output, consumption and investment are negatively serially correlated. The mean capital stock is about 10% above steady state capital.

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[6] I set $\pi_t = 0.5$ when $Z_t \in [\alpha\beta + \Delta, 0.36]$ and $\pi_t = 1 - 10^{-100}$ when $Z_t > 0.36$.

An important shortcoming of the model versions discussed so far is that, when bubble growth stops, then the investment ratio immediately jumps back to the constant $Z^L = \alpha\beta + \Delta$. This generates the abrupt investment and GDP collapses in model variant with a constant probability π. It thus seems interesting to also consider a model version with more gradual investment contractions.

2.1.2. Specification II: gradual contractions

Specification II postulates that $Z^L_{t+1} = \alpha\beta + 0.95 \times (Z_t - \alpha\beta)$. When the expanding phase of a bubble finishes, the investment/output ratio thus falls less than under specification I (discussed above). Until the start of a new bubble expansion, the investment ratio contracts gradually. Panel (a) of Table 3 plots Z^H_{t+1} and Z^L_{t+1}, as a function of $Z_t \in [\alpha\beta, 1)$, under specification II and a constant probability $\pi_t = 0.5$. Simulated paths, generated under the assumption of a constant probability $\pi_t = 0.5$, now generate markedly less abrupt fluctuations of output and investment. The predicted standard deviations of (HP filtered) logged output and investment are 1.73% and 4.94%, respectively. These predicted statistics are broadly in line with the data. However, a limitation of this specification is that the predicted investment/output ratio (0.96) share is now close to unity. Accordingly, the consumption/output share (0.04) is close to zero, and the standard deviation of *logged* consumption (210.3%) is excessive.

A more realistic mean investment ratio, and more realistic consumption volatility are generated if the probability π_t increases once the investment/GDP ratio rise above a threshold. I again consider a setup in which π_t is very close to unity, for values of the investment/output ratio greater than 0.36. Simulated sample paths for that model variant are shown in Panel (c) of Fig. 3. This variant generates a mean investment ratio of 0.37, and a consumption/output ratio of 0.63. The sample paths exhibit recurrent brief phases during which investment and output rise rapidly, while consumption falls. Investment and output decline gradually after investment spurts. The intuition for this is that, after the 0.36 threshold for the investment ratio has been crossed, a long string of 'Low' investment ratios is required to bring the investment/GDO ratio back below the 0.36 threshold. After that adjustment, a new growth spurt can take place. [7] Overall, this model version generates more realistic business cycle statistics than the previous versions. In particular, the predicted standard deviations of output (1.40%), consumption (1.30%) and investment are broadly in line with the data, and output and investment are highly serially correlated. Consumption is now slightly positively correlated, but continues to show too little persistence (autocorrelation 0.28).

These numerical experiments suggest that a very simple RBC model driven by a stationary bubble shows promise for capturing basic business cycle statistics. The key ingredient for quantitative success is that bubbles do not grow too big.

[7] Assume that the date t investment ratio is exactly equal to the threshold: $Z_t = 0.36$; with probability 0.5, the investment ratio in the following period is then $Z^H_{t+1} = 0.4068$, which is markedly above the 0.36 threshold; that 'High' realization of the investment ratio has to be followed by a string of 29 'Low' realizations, governed by $Z^L_{\tau+1} = Z^L_{\tau+1} = \alpha\beta + 0.95 \times (Z_\tau - \alpha\beta)$, until the investment ratio falls again below the 0.36 threshold.

2.2. Transversality condition and OLG population structures

Most of the canonical models discussed in this paper are usually presented as the solution of the decision problems of infinitely-lived representative agents. Those decision problems are well-behaved concave programming problem, and they thus have a unique solution. A necessary optimality condition of an infinitely-lined agent is the transversality condition (TVC) that requires that the value of the capital stock is zero, at infinity: $\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=0$. This implies that the sunspot equilibria considered here violate the TVC of the infinitely-lived agent. Note that, in the Long-Plosser model, $u'(C_t)K_{t+1}=K_{t+1}/C_t=Z_t/(1-Z_t)$. When Z_t approaches unity, this term takes very large values, which causes a violation of the TVC. The TVC is violated even when the probability that the investment ratio 'Z' approaches unity is set at very small values (as in the model versions in which the probability of an investment crash π_t is set at values very close to unity).

This paper abstracts from the TVC. Several theoretical analyses of bubble postulate that market participants perceive the TVC violations as irrelevant, as these violations may pertain to events with a very small probability that occur in a very distant future. See, e.g., Lansing (2010), who disregards the TVC in a Lucas-style asset pricing models with bubbles. arguing that "agents are forward-looking but not to the extreme degree implied by the transversality condition" (Lansing (2010), p.1157).

In richer models, with multiple agents and distortions, the market equilibrium cannot be represented as the optimum of the decision problem of a representative agent, and one would have to resort to numerical simulations to detect TVC violations. However, detection of TVC violations may be extremely difficult, even when very long simulation runs are considered, as histories for which the capital stock explodes and/or consumption approaches zero may occur with extremely small probabilities.

An OLG model with efficient intergenerational risk sharing

Another approach to the TVC issue is to assume that there is no TVC, because the economy is inhabited by overlapping generations (OLG) of finitely-lived agents. I show now that an OLG model can have the same aggregate dynamic equations--except the TVC-- as an economy with an infinitely lived representative agent (see the Appendix for further details).

The two key ingredients for this result are: (i) The assumption of efficient risk sharing between periods t and t+1, among all agents who are alive in both periods; (ii) The assumption that newborn agents receive a wealth endowment such that consumption by newborns represents a time-invariant share of aggregate consumption.

Assume that agents live N<∞ periods, and that a fraction 1/N of the population is aged n=1,…,N. All agents have log utility and the same subjective discount factor, β. Let $c_{i,t}$ denote the date t consumption of agents who are in the i-th period of their life ('generation i') at date t. Aggregate consumption at date t is $C_t=\sum_{i=1}^{N}c_{i,t}$. Assume that there exists a market at date t in which a complete set of state-continent date t+1 payouts is traded. This implies that, in equilibrium, the consumption growth rate between t and t+1 is equated across all agents who are alive in both periods (risk sharing):

$$c_{i+1,t+1}/c_{i,t}=c_{2,t+1}/c_{1,t} \text{ for i=1,..,N-1.} \tag{11}$$

Let $\lambda_{i,t} \equiv c_{i,t}/C_t$ denote the date t consumption share of generation i. Assume that the wealth endowment of the newborn generation sustains a time-invariant consumption share of newborns: $\lambda_{1,t} = \lambda_1 \ \forall t$. When the period utility function is logarithmic (as assumed here), this requires that the wealth endowment of newborns is a time-invariant fraction of aggregate wealth (see the Appendix for proofs of this and the following results).[8] A constant consumption share of newborn agents implies that *each* generation's consumption share is likewise time-invariant: $\lambda_{i,t} = \lambda_i \ \forall t$, and that the capital Euler equation in the OLG economy is (see Appendix):

$$E_t \widetilde{\beta} C_t / C_{t+1} r_{K,t+1} = 1, \text{ with } \widetilde{\beta} \equiv \beta \times (1-\lambda_N)/(1-\lambda_1). \qquad (12)$$

We thus see that, up to a rescaling of the subjective discount factor when $\lambda_1 \neq \lambda_N$, this OLG model implies the same Euler equation, in terms of aggregate consumption, as a model with an infinitely lined agent. If the initial wealth endowment of newborns is such that $\lambda_1 = 1/N$, then $\lambda_i = 1/N$ holds for i=1,..,N, which implies $\widetilde{\beta} \equiv \beta$. In the special case where $\lambda_1 = 1/N$, the capital Euler equation of the OLG economy is thus *identical* to the Euler equation of an economy with an infinitely-lived agent. Assume that the aggregate resource constraint is identical across an OLG economy and an economy with an infinitely lived agent. The only difference between the two economies is that the transversality condition $\lim_{\tau \to \infty} \beta^\tau E_t u'(C_{t+\tau}) K_{t+\tau+1} = 0$ does not hold in the OLG economy, as there is no infinitely-lived agent in the OLG economy. The OLG structure with efficient intergenerational risk sharing thus provides a motivation for considering business cycle models that lack a TVC, but whose other equilibrium conditions are the same as those of standard business cycle models with infinitely lived agents.

2.3. RBC model: incomplete capital depreciation, variable labor

I next show how a bubble equilibrium can be constructed for a more complicated economy: a RBC model with less than 100% capital depreciation and variable labor. A representative household maximizes life-time utility $E_0 \sum_{t=0}^{\infty} \beta^s u(C_t, L_t)$ subject to the resource constraint $C_t + I_t = Y_t$ with $I_t = K_{t+1} - (1-\delta) K_t$ and $Y_t = \theta_t (K_t)^\alpha (L_t)^{1-\alpha}$. C_t, L_t, I_t, K_t and Y_t are consumption, hours worked, (gross) investment and output, respectively at date t. $\theta_t > 0$ is TFP. $0 < \beta, \alpha < 1$ are the subjective discount factor and the elasticity of output with respect to capital (capital share). The period utility function has the form suggested by Greenwood, Hercowitz and Huffman (1988): $u(C_t, L_t) = (1-\sigma)^{-1} (C_t - \nu(L_t))^{1-\sigma}$, $\sigma > 0$, where $\nu(L_t) < C_t$ is an increasing function of hours, $\nu' > 0$. I assume that $\nu(L_t) = (\Psi/(1+1/\eta)) \{L_t^{1+1/\eta} - L^{1+1/\eta}\}$, with $\Psi, \eta > 0$; L denotes steady state hours worked. Assume that TFP follows a stationary AR(1) process: $\ln(\theta_{t+1}) = \rho \ln(\theta_t) + \varepsilon_{t+1}^\theta$ where ε_{t+1}^θ is a white noise innovation and $0 \leq \rho < 1$.

The household's decision problem has these first –order conditions:

[8] Assume that the young can appropriate a constant share of total wealth, perhaps via a political process (or by theft from older agents). The wealth endowment might be provided by a government transfer to newborns financed by a (lump sum) tax levied on older generations. In reality, all advanced societies make significant transfers to young generations (e.g. via subsidized healthcare and education). Wealthier countries make bigger transfers to the young than poorer countries. Thus, it seems reasonable to assume that the wealth endowment of the young is a (roughly) constant share of total wealth.

$$E_t\beta\{(C_{t+1}-\nu(L_{t+1}))^{-\sigma}/(C_t-\nu(L_t))^{-\sigma}\}(\alpha Y_{t+1}/K_{t+1}+1-\delta)=1 \quad \text{and} \tag{13}$$

$$(1-\alpha)Y_t/L_t=\Psi\cdot(L_t)^{1/\eta}. \tag{14}$$

(14) implies that date t hours can be expressed as an increasing function of capital and TFP:

$$L_t=n(K_t,\theta_t)\equiv(\theta_t(K_t^{\alpha})(1-\alpha)/\Psi)^{1/(\alpha+1/\eta)}. \tag{15}$$

The Euler equation (13) implies:

$$\beta\{(C_{t+1}-\nu(L_{t+1}))^{-\sigma}/(C_t-\nu(L_t))^{-\sigma}\}(\alpha Y_{t+1}/K_{t+1}+1-\delta)=1+\varepsilon_{t+1}, \tag{16}$$

where ε_{t+1} is a random variable with zero conditional mean: $E_t\varepsilon_{t+1}=0$. Substituting (15) and the resource constraint $C_t=Y_t+K_t(1-\delta)-K_{t+1}$ into (15) produces the following law of motion for capital:

$$K_{t+2}=\kappa(K_{t+1},K_t,\theta_{t+1},\theta_t,S_{t+1})\equiv$$

$$gy_{t+1}-\nu(n(K_{t+1},\theta_{t+1}))-(gy_t-K_{t+1}-\nu(n(K_t,\theta_t)))(\beta\cdot gmpk_{t+1})^{1/\sigma}\cdot S_{t+1}, \text{ with } S_{t+1}\equiv(1/(1+\varepsilon_{t+1}))^{1/\sigma} \tag{17}$$

where $gy_t=gy(K_t,\theta_t)\equiv Y_t+(1-\delta)K_t$ is period t output plus the depreciated capital stock; $gmpk_{t+1}$ is the gross return on capital in period t+1: $gmpk_{t+1}=\alpha(K_{t+1})^{\alpha-1}(N_{t+1})^{1-\alpha}+1-\delta$. Note that $gy_t-K_{t+1}-\nu(n(K_t,\theta_t))=C_t-\nu(N_t)>0$. Therefore K_{t+2} is an increasing and strictly concave function of ε_{t+1}.

2.3.1. Intuition about the structure of sunspot equilibrium: linearized law motion

The law of motion of the capital stock (17) is more complicated than the law of motion of capital in the Long-Plosser model. To form an idea about the structure of a sunspot equilibrium, I linearize (17) w.r.t $K_{t+1},K_t,\theta_{t+1},\theta_t$ and $S_{t+1}\equiv(1/(1+\varepsilon_{t+1}))^{1/\sigma}$ around the deterministic steady state:

$$dK_{t+2}=\kappa_1 dK_{t+1}+\kappa_2 dK_t+\kappa_3 d\theta_{t+1}+\kappa_4 d\theta_t+\kappa_5 dS_{t+1}, \tag{18}$$

where $dX_t\equiv(X_t-X)/X$ is the relative deviation of X_t from its steady state value, X.

The conventional no-sunspot solution of this model (that imposes a transversality condition) is described by a stationary policy function $K_{t+1}=\lambda(K_t,\theta_t)$. Linearization of that policy function gives: $dK_{t+1}=\lambda_1 dK_t+\lambda_2 d\theta_t$, with $0<\lambda_1<1,\ 0<\lambda_2$. To understand how the (linearized) dynamics of the sunspot equilibrium differs from that of the no-sunspot equilibrium, I rewrite (18) as

$$dK_{t+2}-[\lambda_1 dK_{t+1}+\lambda_2 d\theta_{t+1}]=(\kappa_1-\lambda_1)(dK_{t+1}-[\lambda_1 dK_t+\lambda_2 d\theta_t])+$$

$$(\kappa_2+\lambda_1(\kappa_1-\lambda_1))dK_t+(\kappa_3-\lambda_2)d\theta_{t+1}+(\kappa_4+\lambda_2(\kappa_1-\lambda_1))d\theta_t+\kappa_5 dS_{t+1}. \tag{19}$$

Note that the recursion (17) holds when K_{t+1},K_{t+2} are set according to the no-sunspot policy rule, i.e. when $K_{t+1}=\lambda(K_t,\theta_t)$ and $K_{t+2}=\lambda(K_{t+1},\theta_t)=\lambda(\lambda(K_t,\theta_t),\theta_{t+1})$. Substituting these expressions into (17) gives:

$$\lambda(\lambda(K_t,\theta_t),\theta_t^{\rho}\cdot\exp(\varepsilon_{t+1}^{\theta}))=\kappa(\lambda(K_t,\theta_t),K_t,\theta_t^{\rho}\cdot\exp(\varepsilon_{t+1}^{\theta}),\theta_t,S_{t+1}), \tag{20}$$

where I used the fact that $\theta_{t+1}=\theta_t^{\rho}\cdot\exp(\varepsilon_{t+1}^{\theta})$. (20) has to hold for any values of K_t and of θ_t. Thus, the partial derivatives of the left-hand side of (20) w.r.t. K_t and θ_t have to equal the corresponding derivatives of the right-hand side. Evaluating these derivatives at the steady state gives:

$$(\kappa_2+\lambda_1(\kappa_1-\lambda_1))=0 \text{ and } (\kappa_3-\lambda_2)\rho+(\kappa_4+\lambda_2(\kappa_1-\lambda_1))=0.\ ^{9} \quad (21)$$

The restrictions (21) allow to write (19) as:

$$dK_{t+2}-[\lambda_1 dK_{t+1}+\lambda_2 d\theta_{t+1}]=(\kappa_1-\lambda_1)(dK_{t+1}-[\lambda_1 dK_t+\lambda_2 d\theta_t])+(\kappa_3-\lambda_2)\varepsilon^{\theta}_{t+1}+\kappa_5 dS_{t+1}. \quad (22)$$

(22) describes the dynamics of the gap between the capital stock in the sunspot equilibrium, and the capital stock that would be chose according to the no-sunspot decision rule. Denote that 'capital gap' as $g_t \equiv dK_{t+1}-[\lambda_1 dK_t+\lambda_2 d\theta_t]$. Thus,

$$g_{t+1}=(\kappa_1-\lambda_1)g_t+(\kappa_3-\lambda_2)\varepsilon^{\theta}_{t+1}+k_5 dS_{t+1}. \quad (23)$$

It can be shown that $(\kappa_1-\lambda_1)>1,\ (\kappa_3-\lambda_2)>0,\ k_5<0$. Note that $dS_{t+1}\equiv(1/(1+\varepsilon_{t+1}))^{1/\sigma}-1$. g_{t+1} is hence an increasing and strictly concave function of the sunspot shock ε_{t+1}. This implies the capital gap can be stationary, despite the fact that the linearized law of motion of the 'capital gap' has a root that exceeds unity.

If the gap equals zero, $g_t=0$, then the capital gap may become negative in t+1 if $\varepsilon^{\theta}_{t+1}<0$ and/or $\varepsilon_{t+1}>0$. The strict concavity in ε_{t+1} and $(\kappa_1-\lambda_1)>1$, imply that a negative realization of the capital gap, $g_t<0$, may be followed by an explosive trajectory in negative territory. Therefore, the capital gap in a stationary sunspot equilibrium has to stay strictly positive: $g_t>0$.

I now construct a stationary solution of (23) for which the capital gap never falls below a strictly positive number, $\Delta>0$. This requires that the TFP innovation has a lower bound. For simplicity, assume that $\varepsilon^{\theta}_{t+1}=\pm\sigma_{\varepsilon^{\theta}}$ with probability 0.5, where $\sigma_{\varepsilon^{\theta}}$ is the standard deviation of $\varepsilon^{\theta}_{t+1}$. To ensure existence of an equilibrium with $g_t\geq\Delta\ \forall t$, Δ has to be set sufficiently big relative to $\sigma_{\varepsilon^{\theta}}$. Specifically, we need that

$$(\kappa_1-\lambda_1)\Delta+(\kappa_3-\lambda_2)\cdot(-\sigma_{\varepsilon^{\theta}})>\Delta, \text{ i.e. } \Delta>\{(\kappa_3-\lambda_2)/(\kappa_1-\lambda_1-1)\}(-\sigma_{\varepsilon^{\theta}}). \quad (24)$$

As in the previous analyses, I assume that ε_{t+1} takes only two values, $-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\pi/(1-\pi)$, with probabilities π and $1-\pi$, respectively, for $\overline{\varepsilon_t}\in[0,1]$. For given values of g_t and $\varepsilon^{\theta}_{t+1}$, we thus see that g_{t+1} takes two values: $g_{t+1}\in\{g^L_{t+1},g^H_{t+1}\}$. I set $\overline{\varepsilon_t}$ such that

$$\Delta=(\kappa_1-\lambda_1)g_t+(\kappa_3-\lambda_2)\varepsilon^{\theta}_{t+1}+k_5\cdot\{(1/(1-\overline{\varepsilon_t}))^{1/\sigma}-1\}. \quad (25)$$

Therefore,

$$g^L_{t+1}=\Delta \text{ and } g^H_{t+1}=(\kappa_1-\lambda_1)g_t+(\kappa_3-\lambda_2)\varepsilon^{\theta}_{t+1}+k_5\cdot\{(1/(1+\overline{\varepsilon_t}\cdot\pi/(1-\pi)))^{1/\sigma}-1\}. \quad (26)$$

Solving (24) for $\overline{\varepsilon_t}$ gives:

$$\overline{\varepsilon_t}=1-[1+\{(k-\lambda_1)g_t+(\kappa_3-\lambda_2)\varepsilon^{\theta}_{t+1}-\Delta\}/(-k_5)\ ]^{-\sigma}, \quad (27)$$

with $0<\overline{\varepsilon_t}<1$, as $(k-\lambda_1)g_t+(\kappa_3-\lambda_2)\varepsilon^{\theta}_{t+1}-\Delta>0$. As $k_5\cdot\{(1/(1+\overline{\varepsilon_t}\cdot\pi/(1-\pi)))^{1/\sigma}-1\}>0$, we see that

$$g^L_{t+1}\leq g_t<g^H_{t+1}\leq(\kappa_1-\lambda_1)g_t+(\kappa_3-\lambda_2)\sigma_{\varepsilon^{\sigma}}+k_5((1-\pi)^{1/\sigma}-1). \quad (28)$$

With probability $1-\pi$, the period t+1 'capital gap' increases, thus, relative to the 'gap' g_t in period t. With probability π, the 'gap' at t+1 is $g^L_{t+1}=\Delta$. Thus,

[9] The slope coefficients λ_1,λ_2 of the no-sunspots decision rule can be determined as functions of $\kappa_1,\kappa_2,\kappa_3,\kappa_4$ and ρ by solving (21).

$$E_t g_{t+1} = \pi g^L_{t+1} + (1-\pi) E_t g^H_{t+1} \leq \pi\Delta + (1-\pi)\{(\kappa_1 - \lambda_1) g_t + (\kappa_3 - \lambda_2)\sigma_{\varepsilon^\sigma} + k_5((1-\pi)^{1/\sigma} - 1)\}.$$

A sufficient condition for $\lim_{s\to\infty} E_t g_{t+s} < \infty$ is thus $(1-\pi)(\kappa_1 - \lambda_1) < 1 \Leftrightarrow \pi > 1 - 1/(\kappa_1 - \lambda_1)$. For plausible calibrations, $\kappa_1 - \lambda_1$ is only slightly greater than unity, and thus a small 'crash' probability π suffices to ensure that the capital gap does not explode. (For the calibration discussed in the next paragraph, π>3.99% is needed.) Provided this restriction is met, we have

$$\lim_{s\to\infty} E_t g_{t+s} < \{\pi\Delta + (\kappa_3 - \lambda_2)\sigma_{\varepsilon^\sigma} + k_5((1-\pi)^{1/\sigma} - 1)\} / [1 - (1-\pi)(\kappa_1 - \lambda_1)].$$

Fig 4 shows numerical simulations for the parameter values: $\beta = 0.99, \alpha = 0.35, \sigma = 1, \eta = 1,$ $\delta = 0.025$. Δ is set at 1% of the steady state capital stock, K: $\Delta = 0.01 \times K$. These parameter values imply the following slope coefficients of the linearized law of motion of the capital gap: $\kappa_1 - \lambda_1 = 1.0416$, $k_3 = -0.0753$. To focus on the effect of sunspot shocks, I set the TFP innovations at zero: $\sigma_{\varepsilon^\theta} = 0$. The law of motion of the capital gap is thus given by:

$$g_{t+1} = \gamma(g_1, \varepsilon_{t+1}) \equiv (\kappa_1 - \lambda_1) g_t + k_5\{(1/(1+\varepsilon_{t+1}))^{1/\sigma} - 1\}. \quad (29)$$

The parameters of the sunspot process are set at $\Delta = 0.01$, $\pi = 0.5$. Fig. 4a plots g^L_{t+1}, g^H_{t+1} and $E_t g_{t+1} = \pi g^L_{t+1} + (1-\pi) g^H_{t+1}$, as functions of g_t, while Fig.4b shows a simulated path of $\{g_t\}$ over 10000 periods. Also plotted is $g_{t+1} = \gamma(g_t, 0)$, i.e. the (explosive) law of motion of the 'capital gap' in the absence of stochastic sunspot shocks. As shown in Fig 4a, $g^L_{t+1} < \gamma(g_t, 0) < g^H_{t+1}$ for $g_t > \Delta$. Note also that $g^U_{t+1} - g_t \ll g_t - g^L_{t+1}$. The calibration here implies that $E_t g_{t+1} < g_t$ and that $\partial E_t g_{t+1} / \partial g_t < 1$ hold for $g_t > 0.018$. The process has strong mean reversion, due to the concavity of $g_{t+1} = \gamma(g_t, \varepsilon_{t+1})$ with respect to ε_{t+1}. The simulated sample path shows that the capital gap stays mostly close to the lower bound Δ, but that it occasionally diverges from the vicinity of Δ, before abruptly 'crashing' to Δ.

2.3.2. Stationary sunspots in the non-linear RBC model

The previous discussion has set the stage for the construction of stationary sunspots equilibria in the non-linear RBC model (17). I continue to focus on economies without TFP shocks, and assume that $\theta_t = 1 \ \forall t$. Given K_t, K_{t+1}, I assume that K_{t+2} takes two values $K^L_{t+2} \leq K^H_{t+2}$ with probabilities π and $1-\pi$, respectively. I set the 'Low' realization of the date t+1 sunspot shock, i.e. $\varepsilon_{t+1} = -\overline{\varepsilon_t} \leq 0$ such that it induces a value of K^L_{t+2} that equals the capital stock dictated by the no-sunspot policy rule, $\lambda(K_{t+1}, \theta_t)$, plus a positive constant Δ. Thus, $\overline{\varepsilon_t}$ is defined by

$$K^L_{t+2} \equiv \lambda(K_{t+1}, \theta_t) + \Delta = \kappa(K_{t+1}, K_t, \theta_{t+1}, \theta_t, -\overline{\varepsilon_t}). \quad (30)$$

Solving this equation for $\overline{\varepsilon_t}$ gives:

$$\overline{\varepsilon_t} = 1 - \beta \cdot gmpk_{t+1} \cdot \{(gy_t - K_{t+1} - \nu(n(K_t, \theta_t))) / (gy_{t+1} - [\lambda(K_{t+1}, \theta_t) + \Delta] - \nu(n(K_{t+1}, \theta_{t+1}))\}^\sigma. \quad (31)$$

Substituting $\varepsilon_{t+1} = \overline{\varepsilon_t}\pi/(1-\pi)$ into (17) defines the 'High' value of the t+2 capital stock:

$$K^H_{t+2} \equiv \kappa(K_{t+1}, K_t, \theta_{t+1}, \theta_t, \overline{\varepsilon}_t \pi_t/(1-\pi_t))).^{10} \tag{32}$$

Simulation results are reported in Table 2 and in Figure 5. In order to focus on the role of sunspot shocks, I again abstract from TFP shocks. I use the same parameter values as above. The no-sunspot policy rule for capital, $\lambda(K_{t+1}, \theta_t)$, is approximated using a second-order Taylor expansion. The predicted standard deviations of HP filtered logged GDP, consumption, investment and hours are 0.46%, 9.67%, 11.12% and 0.23%, respectively. Thus, consumption and investment are excessively volatile. Fluctuations of GDP, consumption and investment induced by bubble shocks are smaller in the RBC model (incomplete capital depreciation) than in baseline the Long-Plosser model. The more muted fluctuations of GDP are due to the fact that the capital stock is less volatile under incomplete depreciation. As in other models driven by investment demand shocks, consumption and investment are negatively correlated. Consumption is predicted to be weakly procyclical, while investment is predicted to be weakly countercyclical. This is due to the fact that, after the burst of a bubble, output and consumption remain elevated for some time and decay gradually, while investment contracts.

2.3.3. Tractable numerical solution method for bubbles equilibrium

The approximate law of motion for capital used in Section 2.3.1. matches closely the fully non-linear solution discussed in Sect. 2.3.2. The key feature of the approach in Sect. 2.3.1. is that a linear approximation of the law of motion of capital is taken with respect to capital, TFP and $S_{t+1} \equiv ((1/(1+\varepsilon_{t+1}))^{1/\sigma} - 1$, but *not* w.r.t. ε_{t+1}. This approach preserves the non-linear effect of the sunspot ε_{t+1} on the capital stock. This approach might be useful for handling larger models.

2.4. Small open economy model

Consider a small open economy whose representative household receives an exogenous endowment Y_t of a homogeneous output/consumption good in period *t*. The household can lend or borrow in the world capital market using a one-period bond. The country's period t budget constraint is

$$C_t + A_{t+1} = A_t R_t + Y_t, \tag{33}$$

where C_t is consumption, while A_t denotes net foreign assets (NFA) at the end of period t-1; $R_t > 0$ is the gross interest rate on that NFA position. Y_t denotes exogenous output. Y_t follows an AR(1) process. The household's period utility function is $u(C_t) = (C_t^{1-\sigma} - 1)/(1-\sigma)$, where $\sigma > 0$ is the coefficient of relative risk aversion. The household's Euler equation is

$$\beta E_t (C_{t+1}/C_t)^{-\sigma} R_{t+1} = 1. \tag{34}$$

[10] Equation (23) gives a positive value $\overline{\varepsilon}_t \geq 0$, provided Δ is set at a sufficiently large positive value (relative to the absolute value of the TFP innovation, $\sigma_{\varepsilon^\theta}$. When there are no TFP shocks, then (23) produces $\overline{\varepsilon}_t \geq 0$ for any $\Delta > 0$. Note that ensuring $\overline{\varepsilon}_t \geq 0$ is crucial for generating a stationary sunspot equilibrium. $\overline{\varepsilon}_t < 0$ would imply $K^U_{t+2} < K^L_{t+2}$, which can trigger an explosive trajectory in which the capital stock falls further and further below the value dictated by the no-sunspot decision rule (see discussion of the linearized dynamics).

To ensure the existence of a stationary equilibrium, it is assumed that R_t is a decreasing function of the country's NFA, due to transaction costs or other frictions: $R_t=R(A_t)$, with $R'<0$. [11] However, the household acts like a price take in the global bond market.

This small open economy model resembles the closed economy RBC models discussed above, as it also assumes one asset, whose return is a decreasing function of the asset stock. However, a plausible calibration implies that, in the open economy model, the asset return is much less sensitive to the asset stock (see below). An interesting further difference is that the asset stock in the open economy can take negative values.

The conventional no-sunspot solution of this model is described by a stationary policy function $A_{t+1}=\lambda(A_t,Y_t)$. Here we construct stationary bubbles that entail deviations from the no-sunspots decision rule.

Euler equation (34) implies $\beta(C_{t+1}/C_t)^{-\sigma}R_{t+1}=1+\varepsilon_{t+1}>0$, where ε_{t+1} is a random variable of zero conditional mean, $E_t\varepsilon_{t+1}=0$. Thus $C_{t+1}=C_t\cdot\{\beta\cdot R(A_{t+1})/(1+\varepsilon_{t+1})\}^{1/\sigma}$ holds. Substituting consumption from the budget constraint into this equation gives the following law of motion for NFA:

$$A_{t+2}=\kappa(A_{t+1},A_t,Y_{t+1},Y_t,\varepsilon_{t+1})\equiv Y_{t+1}+A_{t+1}R(A_{t+1})-(Y_t+A_tR(A_t)-A_{t+1})\{\beta R(A_{t+1})/(1+\varepsilon_{t+1})\}^{1/\sigma}. \qquad (35)$$

Note that A_{t+2} is an increasing and strictly concave function of ε_{t+1} (for positive values of date t consumption, $C_t=Y_t+A_tR(A_t)-A_{t+1}$). This again allows to construct a stationary sunspot equilibrium.

As in previous models, I assume that ε_{t+1} takes two possible values: $\varepsilon_{t+1}=-\overline{\varepsilon_t}\leq 0$ and $\varepsilon_{t+1}=\overline{\varepsilon_t}\pi_t/(1-\pi_t)$, with probabilities π_t and $1-\pi_t$, respectively. Conditional on A_{t+1},A_t,Y_{t+1} and Y_t, the end-of-period t+1 NFA takes thus two values, $A^L_{t+2}\leq A^H_{t+2}$. I set the 'Low' realization of $\varepsilon_{t+1}=-\overline{\varepsilon_t}$ in such a manner that A^L_{t+2} equals the NFA dictated by the no-sunspots decision rule, plus a positive constant Δ: $\lambda(A_{t+1},Y_t)+\Delta=A^L_{t+2}\equiv\kappa(A_{t+1},A_t,Y_{t+1},Y_t,-\overline{\varepsilon_t})$. Solving this equation for $\overline{\varepsilon_t}$ gives:

$$\overline{\varepsilon_t}=1-\beta R(A_{t+1})\{(Y_t+A_tR(A_t)-A_{t+1})/(Y_{t+1}+A_{t+1}R(A_{t+1})-[\lambda(A_{t+1},Y_t)+\Delta])\}^{1/\sigma}. \qquad (36)$$

Substituting $\varepsilon_{t+1}=\overline{\varepsilon_t}\pi/(1-\pi)$ into (35) defines the 'High' value of the t+2 NFA: $A^H_{t+2}\equiv\kappa(A_{t+1},A_t,Y_{t+1},Y_t,\overline{\varepsilon_t}\pi_t/(1-\pi_t))$.

Fig. 5 and Table 3 report numerical simulation results, for the small open economy model. To highlight the effect of sunspots, I assume that output is constant, $Y_t=Y\ \forall t$. As before, I set $\sigma=1$ and $\beta=0.99$. The simulations assume that $R_t=\exp(-aA_t/Y)/\beta$, with $a>0$. Thus, NFA is zero, in the deterministic steady state. I set a=0.01.[12] Δ is set at 1% of steady state output, Y: $\Delta=0.01\cdot Y$. The bubble bust probability is set at $\pi_t=0.5$. The no-sunspot policy rule for NFA,

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[11] This assumption follows, e.g., Turnovsky (1985), Kollmann (2002) and Schmitt-Grohé and Uribe (2003).

[12] Kollmann (2002) reviews empirical evidence about the slope coefficient 'a'. That evidence suggests that, for OECD countries, the slope coefficient 'a is about 0.002, when the foreign interest rate facing a given country is expressed as a function of its net foreign assets normalized by quarterly exports. If the exports/GDP ratio is 0.2, the implied slope coefficient is, thus, a=0.01=0.002/0.2.

$\lambda(A_{t+1}, Y_t)$, is approximated using a second-order Taylor expansion. Simulated sample paths (see Figure 6) show that NFA can experience large sudden increases, that are followed by somewhat less rapid contractions. Therefore, the trade balance and consumption undergo sharp reversals. Simulated moments of HP filtered simulated series indicate that, despite the relatively rapid NFA expansions and contractions, the autocorrelation of NFA is sizable (0.90). The bubbly open economy has a positive mean NFA (of about 24% of steady state output). Accordingly, mean consumption is above steady state consumption, and mean net exports are negative (-0.09% of steady state output). The model generates more muted fluctuations if we assume that the bust probability π_t increases when NFA exceeds a threshold. The model generates a realistic standard deviation of consumption when it is assumed that the bust probability is π_t=0.5 for an NFA/GDP ratio up to 25%, and that the bust probability is very close to unity $\pi_t=1-10^{-100}$ when the NFA/GDP ratio exceeds 25%.

2.5. Two country model

[TBA]

3. Conclusion

Linearized Dynamic Stochastic General Equilibrium (DSGE) models with a unique stable solution are the workhorses of modern macroeconomics. This paper shows that stationary sunspot equilibria exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions. In the sunspot equilibria considered here, the economy may temporarily diverge from the no-sunspots allocation, before abruptly reverting towards that allocation. In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity. The quantitative results presented in this paper suggest that simple non-linear DSGE models driven just by stationary bubbles can capture key business cycle stylized facts.

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Appendix: An OLG version of the Long-Plosser economy

Consider the OLG economy discussed in Sect. 2.2. Agents live for N periods. I refer to agents of age i as 'generation i'. Assume that each generation has the same size (number of members). The period t consumption of generation i is denoted $c_{i,t}$, and aggregate consumption is $C_t=\sum_{i=1}^{N} c_{i,t}$. Let $\lambda_{i,t}=c_{i,t}/C_t$ for i=1,..,N be the ratio of generation i's consumption divided by aggregate consumption, in period t. I refer to $\lambda_{i,t}$ as the 'consumption share' of generation i, in period t.

Assume that there exists a financial market at date t for in which a complete set of state-continent assets with payouts at t+1 are traded. This allows consumption risk sharing between t and t+1, among all agents alive in these periods. Thus, these agents' consumption growth rates between t and t+1 are equal, in equilibrium:

$$\lambda_{i+1,t+1}/\lambda_{i,t}=\lambda_{2,t+1}/\lambda_{1,t} \text{ for i=1,..,N-1.} \tag{A.1}$$

(A.1) and the adding up constraint $\sum_{i=1}^{N}\lambda_{i,t}=1$ can be used to solve for the date t+1 consumption shares $\{\lambda_{i,t+1}\}_{i=1,..,N}$ for given date t shares $\{\lambda_{i,t}\}_{i=1,..,N}$: $\lambda_{i+1,t+1}=\lambda_{i,t}(1-\lambda_{1,t})/(1-\lambda_{N,t})$ for i=1,..,N-1.

Assume that the consumption share of newborn agents, during the first period of their life, is time-invariant: $\lambda_{1,t}=\lambda_1 \ \forall t$. That constant newborn consumption share can be sustained by a suitable choice of wealth endowment of the newborn agents. (If the utility function is logarithmic, as assumed here, the wealth endowment of the newborns in each period is a time-invariant share of total wealth in that period—see below.) When $\lambda_{1,t}=\lambda_1$, then (A.1) is a stable difference equation in the consumption shares, and the consumption shares of generations i=2,..,N converge asymptotically to a constant consumption shares λ_i (numerical experiments show that convergence to the steady state shares is fast). The N steady state consumption shares are defined by

$$\lambda_{i+1}=\lambda_i(1-\lambda_1)/(1-\lambda_N) \text{ for i=1,..,N-1.} \tag{A.2}$$

Given any newborn's consumption share $0<\lambda_1\leq 1$, these equations pin down unique consumption shares of generations i=2,..,N that are consistent with the adding up constraint $\sum_{i=1}^{N}\lambda_i=1$.

The following discussion assumes that the consumption shares equal their steady state values. Then the marginal rate of substitution between consumption at date t and t+1 of the cohort that that has age i=1,..,N-1 in period t is

$$\rho_{i,t,t+1}\equiv\beta c_{i,t}/c_{i+,t+1}=\beta(\lambda_i C_t)/(\lambda_{i+1}C_{t+1})=\beta\{(1-\lambda_N)/(1-\lambda_1)\}C_t/C_{t+1}. \tag{A.3}$$

(A.3) shows that all cohorts that are alive in t and t+1 have the same intertemporal marginal rate of substitution (IMRS), that will henceforth be denotes as $\rho_{t,t+1}$:

$$\rho_{t,t+1}\equiv\widetilde{\beta}C_t/C_{t+1}, \text{ where } \widetilde{\beta}\equiv\beta(1-\lambda_N)/(1-\lambda_1). \tag{A.4}$$

Assume that capital investment decision in period t are made by a competitive firm that discounts the date t+1 return on capital using $\rho_{t,t+1}$. The first-order condition of that firm is $E_t\widetilde{\beta}(C_t/C_{t+1})\cdot r_{K,t+1}=1$, where $r_{K,t+1}$ is the gross marginal product of capital. We thus see that, up to a rescaling of the subjective discount factor when $\lambda_1\neq\lambda_N$, the OLG model with complete intergenerational risk sharing implies the same Euler equation as the model with an infinitely lined agent. (A.2) implies that when the new=born consumption share is set at $\lambda_1=1/N$, then

λ_i=1/N holds for generations i=2,..,N, which implies that $\widetilde{\beta}\equiv\beta$. We thus see that when the consumption share of the new-born generation is 1/N, then the capital Euler equation of the OLG economy is *identical* to the Euler equation of an economy with an infinitely-lived agent.

Wealth shares

A time-invariant consumption share of the new-born cohort is sustained by allocating to newborn agents a time-invariant share of the aggregate wealth of all cohorts. Let $\omega_{i,t}$ denote the wealth of generation i in period t. For i=1,..,N-1, $\omega_{i,t}$ equals the present value of that generation's consumption stream: $\omega_{i,t}=E_t\sum_{s=0}^{N-i}\rho_{t,t+s}c_{i+s,t+s}$, where the stochastic discount factor (SDF) is a product of the one-period-ahead discount factors defined in (A.4): $\rho_{t,t}=1$ and $\rho_{t,t+s}=\Pi_{\tau=1}^{\tau=s-1}\rho_{\tau,\tau+1}$ for $s>1$. Generation i agent equates her intertemporal marginal rate of substitution to the SDF: $\rho_{t,t+s}=\beta^s c_{i,t}/c_{i+s,t+s}$ for $0<s\leq N-i$. Therefore $\omega_{i,t}=c_{i,t}\sum_{s=0}^{N-i}\beta^s$ and thus

$$c_{i,t}=\phi_i\cdot\omega_{i,t},\ \text{with}\quad \phi_i\equiv(1-\beta)/(1-\beta^{N-i+1})\ \text{for i=1,..,N.} \tag{A.4}$$

Thus, in each period, generation i consumes a fraction ϕ_i of her wealth that is generation-specific, but time invariant. In an equilibrium with constant a consumption share of generation *i*, $\lambda_i=c_{i,t}/C_t$, the period t wealth of generation *i* equals thus $\omega_{i,t}=(\lambda_i/\phi_i)C_t$, and the wealth share of that generation is

$$\omega_{i,t}/\sum_{s=1}^{N}\omega_{s,t}=(\lambda_i/\phi_i)/\sum_{s=1}^{N}(\lambda_s/\phi_s)\equiv\kappa_i\,. \tag{A.5}$$

Note that this wealth share is time-invariant. Thus, an equilibrium with time-invariant generational consumption shares exhibits time-invariant generational wealth shares. As pointed out above, the consumption share of newborn generations, λ_1, pins down uniquely the consumption shares of older generations, i.e. λ_i is a function of λ_1: $\lambda_i=\Lambda_i(\lambda_1)$. There is, hence, a unique mapping from λ_1 to the wealth shares of all generations:

$$\kappa_i=\mathrm{K}_i(\lambda_1)=(\Lambda_i(\lambda_1)/\phi_i)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s)\,. \tag{A.6}$$

If the new-born generation is allocated a wealth share $\kappa_1=(\lambda_1/\phi_1)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s)$, then this sustains an equilibrium in which the consumption share of the new-born generation is λ_1. A consumption allocation in which all generations have consumption share λ_i=1/N is sustained by allocating to the newborn generation a wealth share $\kappa_1=(1/\phi_1)/\sum_{s=1}^{N}1/\phi_s$. As an example, assume that life lasts 80 years, i.e. N=320 quarters, and that the quarterly subjective discount factor is β=0.99; then the consumption allocation with equal consumption shares λ_i=1/N=0.3125% requires a newborn wealth share of κ_1=0.4267%.

Table 1. Long-Plosser model with bubbles: predicted business cycle statistics

	Standard dev. %			Corr. with Y		Autocorr.			Mean (% deviation from SS)			
	Y	C	I	C	I	Y	C	I	Y	C	I	Z
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
(a) Specification I: $Z_t^L=\alpha\beta+\Delta$												
π_t=0.5	11.72	100.19	33.48	-0.42	0.62	0.62	0.47	0.62	13.49	-7.62	53.31	31.15
$\pi_t\cong1$ for $z_t>0.36$	1.33	3.51	3.82	0.77	-0.26	-0.26	-0.66	-0.26	3.27	-0.13	9.71	6.25
(b) Specification II: $Z_t^L=\alpha\beta+0.95\times(z_t-\alpha\beta)$												
π_t=0.5	1.73	210.28	4.94	-0.31	0.68	0.68	0.46	0.68	73.09	-89.68	380.11	177.21
$\pi_t\cong1$ for $z_t>0.36$	1.40	1.30	4.00	0.14	0.85	0.85	0.28	0.85	4.45	-0.26	13.34	8.46
(c) US Data (from King and Rebelo (1999))												
	1.81	1.35	5.30	0.88	0.80	0.88	0.80	0.87				

Note: all business statistics pertain to HP-filtered logged variables.

Table 2. RBC model (incomplete capital depreciation) with bubbles: predicted business cycle statistics

Standard dev. %				Corr. with Y			Autocorr.				Mean (% deviation from SS)				
Y	*C*	*I*	*L*	*C*	*I*	*L*	*Y*	*C*	*I*	*L*	*Y*	*C*	*I*	*L*	*K*
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*	*(16)*
0.46	9.67	11.12	0.23	0.13	-0.21	1.00	0.91	0.54	0.52	0.91	18.6	11.8	39.0	8.90	39.0

Table 3. Small Open Economy model (endowments) with bubbles: predicted business cycle statistics

	Standard dev. %			Corr. with Y		Autocorr.			Mean (% deviation from SS)			
	A	C	NX	C	NX	A	C	NX	A	Y	C	NX
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
π_t=0.5	11.43	9.07	4.95	---	---	0.90	0.53	0.52	24.44	0.00	0.09	-0.09
$\pi_t\cong1$ for $A_{t+1}/Y>0.25$	2.07	1.30	1.27	---	---	0.81	0.26	0.26	16.70	0.00	0.13	-0.13

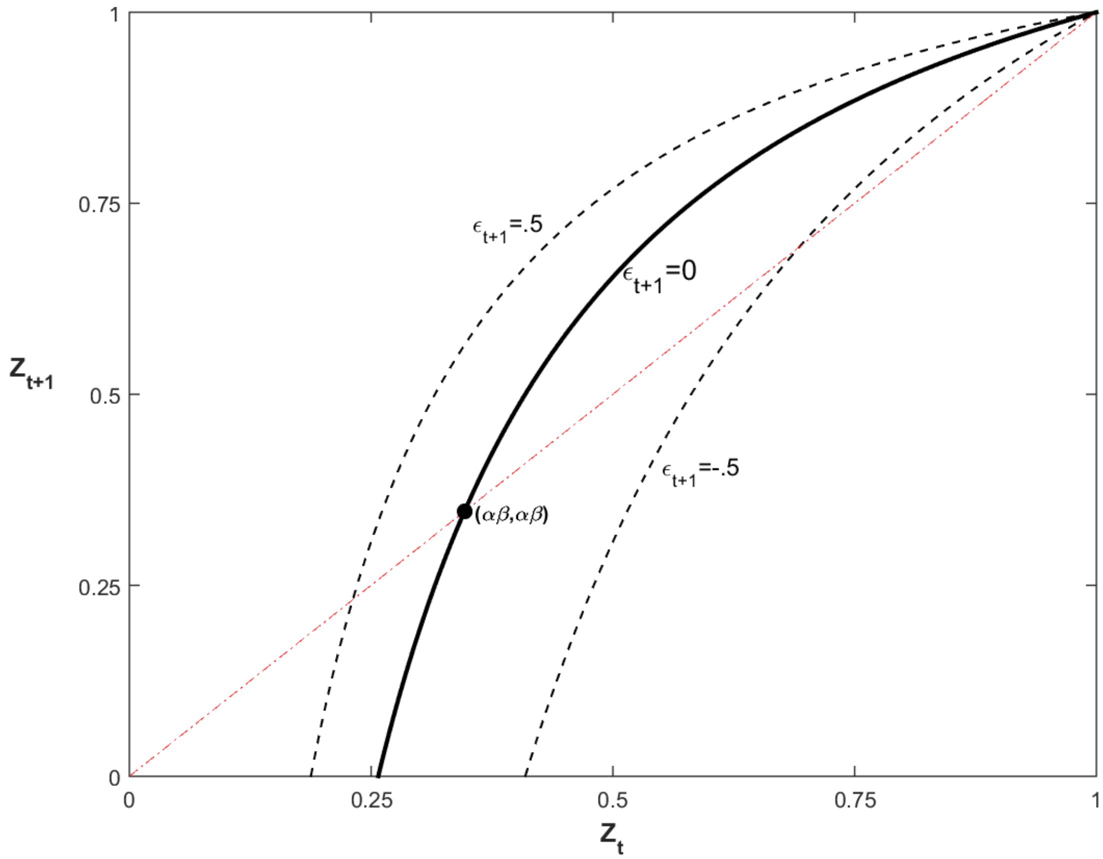
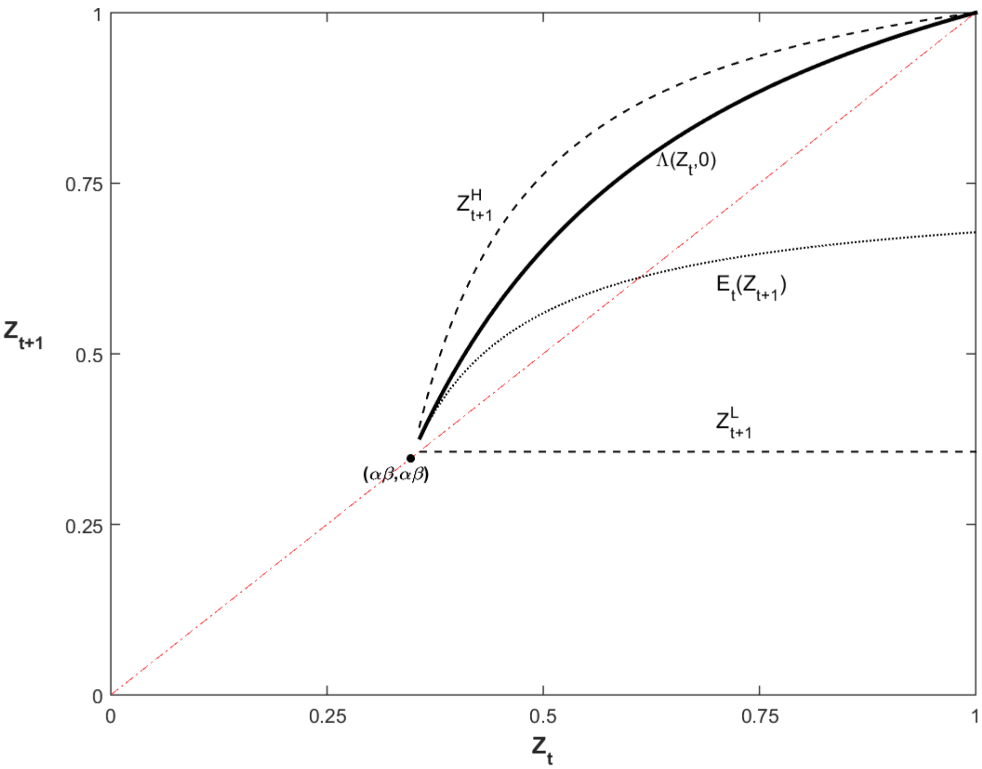
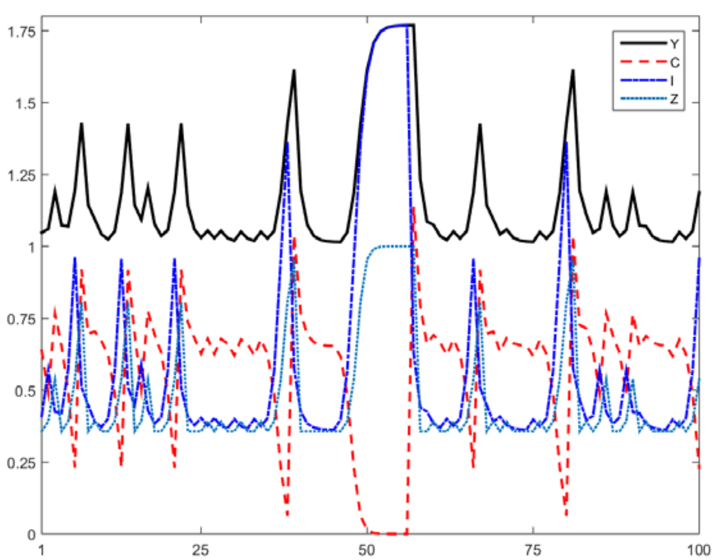


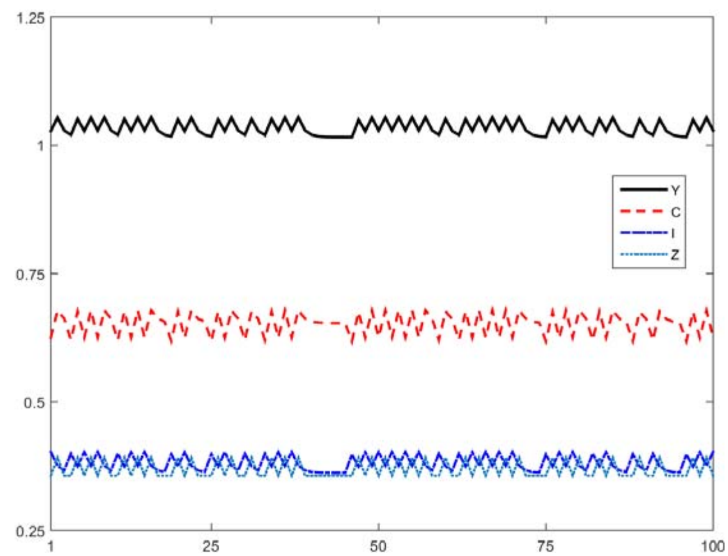
Fig.1. Long & Plosser model: investment/output ratio at t+1, Z_{t+1}, as a function of Z_t for $\varepsilon_{t+1} \in \{-0.5; 0; 0.5\}$



(a) 'Low' and 'High' values of date t+1 investment/output ratio (Z_{t+1}^L, Z_{t+1}^H) and expected value $(E_t Z_{t+1})$ shown as function of $Z_t \in [\alpha\beta+\Delta, 1)$. $\Lambda(Z_t, 0)$ is value of Z_{t+1} without random sunspot. Probability of 'Low' value Z_{t+1}^L: $\pi_t = 0.5 \;\; \forall t \;\; Z_t \in [\alpha\beta+\Delta, 1)$

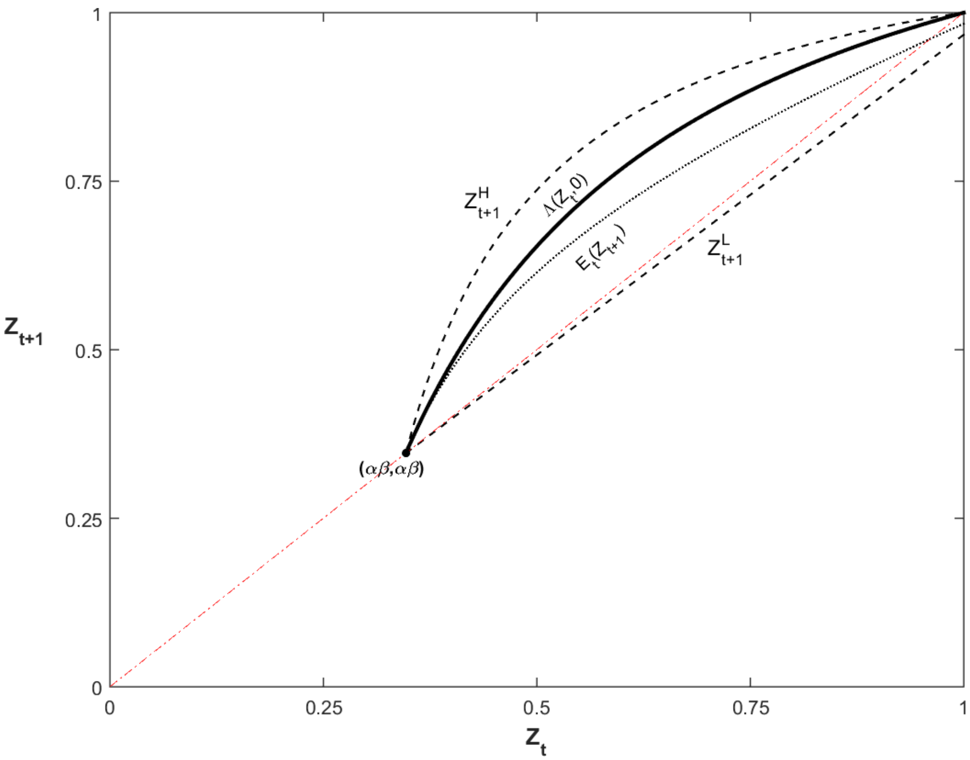


(b) Simulated series with constant probability: $\pi_t = 0.5$.

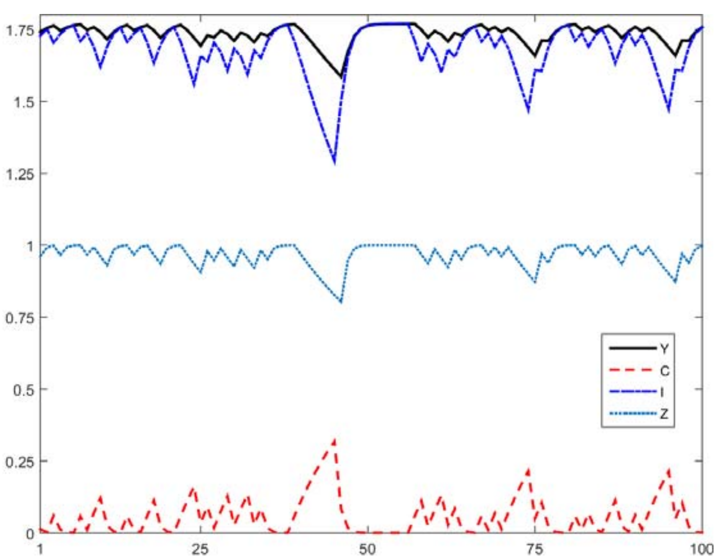


(c) Simulated series with $\pi_t = 0.5$ for $Z_t \leq 0.36$ and $\pi_t \simeq 1$ for $Z_t > 0.36$

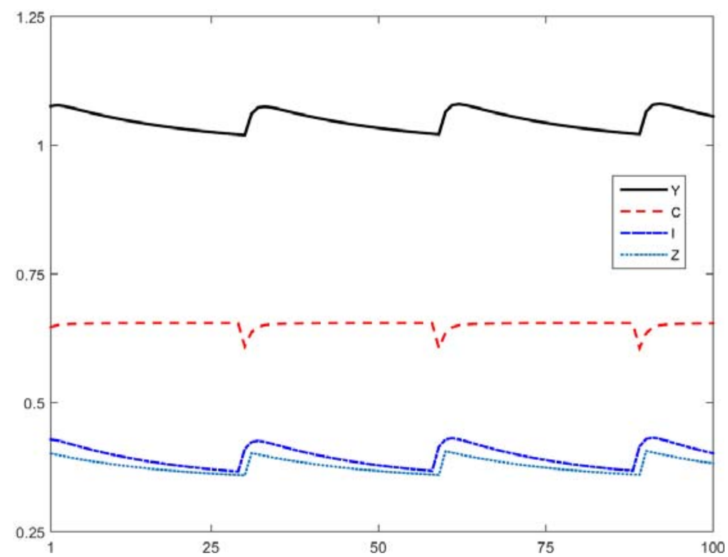
Fig.2. Long & Plosser model with bubbles. Specification I: fast crashes $(Z_{t+1}^L = \alpha\beta + \Delta)$.
Simulated series of output (Y), consumption (C) and investment are normalized by steady state output.



(a) 'Low' and 'High' values of date t+1 investment/output ratio (Z^L_{t+1}, Z^H_{t+1}) and expected value $(E_t Z_{t+1})$, as a function of $Z_t \in [\alpha\beta,1]$. $\Lambda(Z_t,0)$ is value of Z_{t+1} without random sunspot. Probability of 'Low' value Z^L_{t+1}: π_t=0.5 $\forall t$ $Z_t \in [\alpha\beta+\Delta,1)$

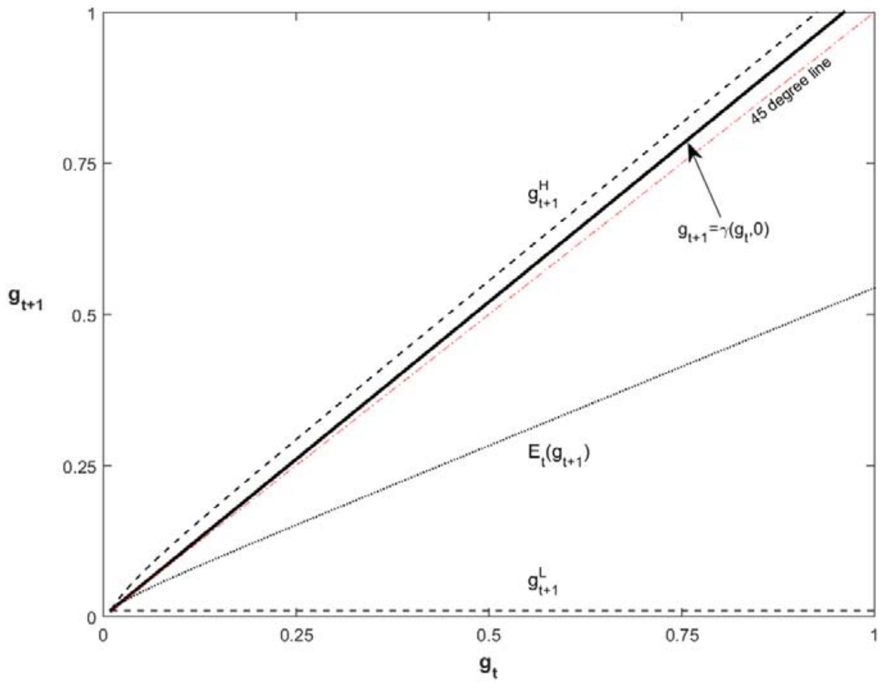


(b) Simulated series with constant probability: π_t=0.5.

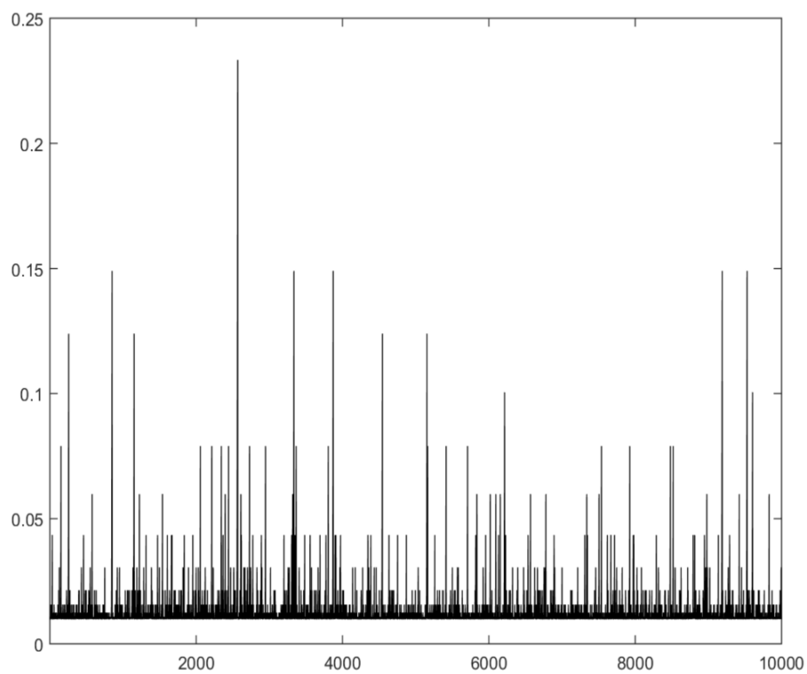


(c) Simulated series with π_t=0.5 for $Z_t \leq 0.36$ and $\pi_t \simeq 1$ for $Z_t > 0.36$

Fig.3. Long & Plosser model with bubbles. Specification II: gradual contractions $(Z^L_{t+1} = \alpha\beta + 0.95 \cdot (Z_t - \alpha\beta))$. Simulated series of output (Y), consumption (C) and investment are normalized by steady state output.



(a) Law of motion of the sunspot 'capital gap'



(b) Simulated sunspot 'capital gap' (constant crash probability: π=0.5)

Fig.4 Linearized RBC model (incomplete capital depreciation, variable labor): dynamics of the sunspot equilibrium 'capital gap'

g_t: difference between capital stock in sunspot equilibrium and capital prescribed by no-sunspot policy rule

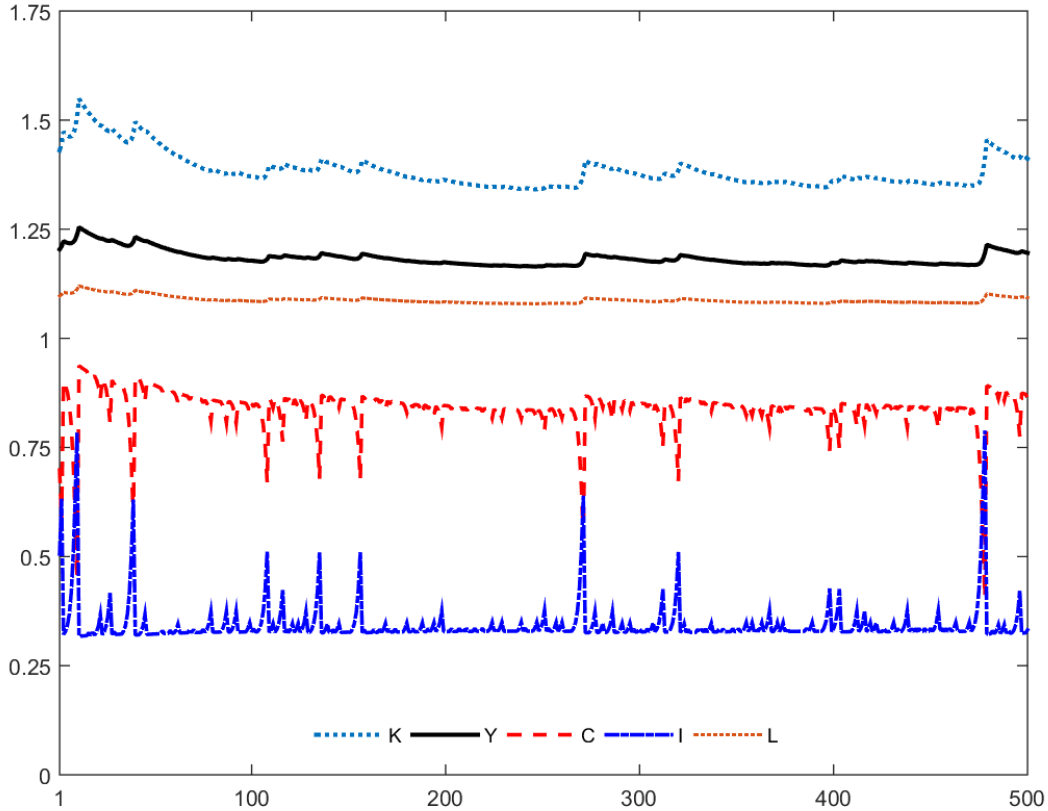
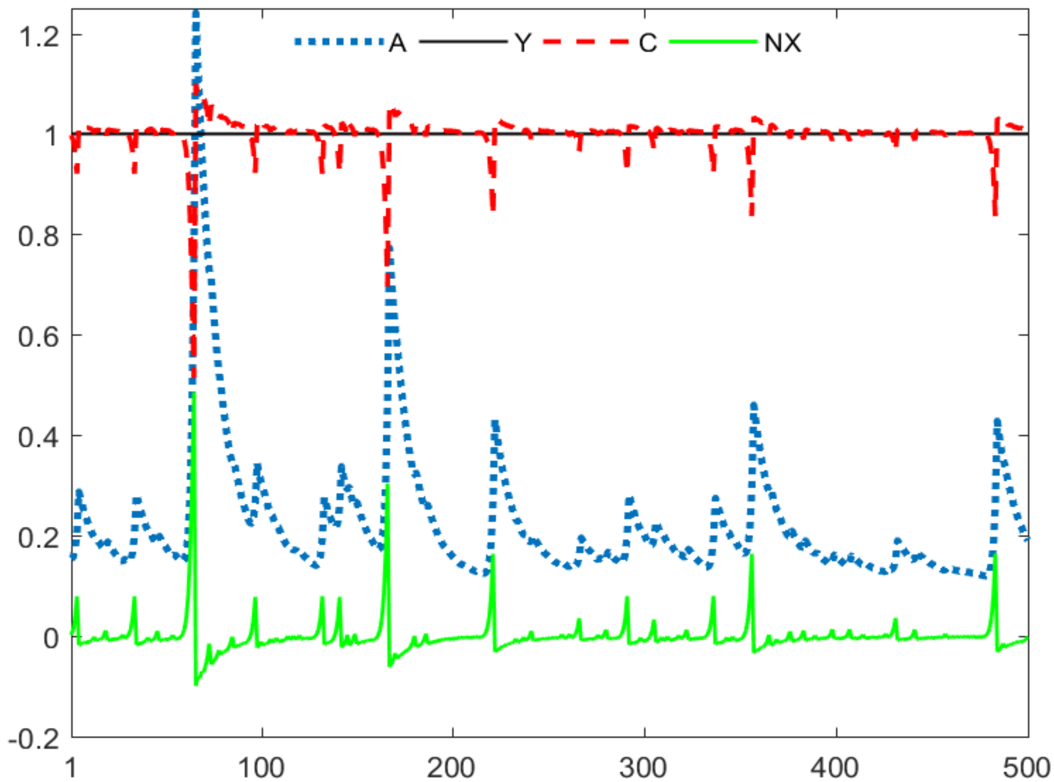
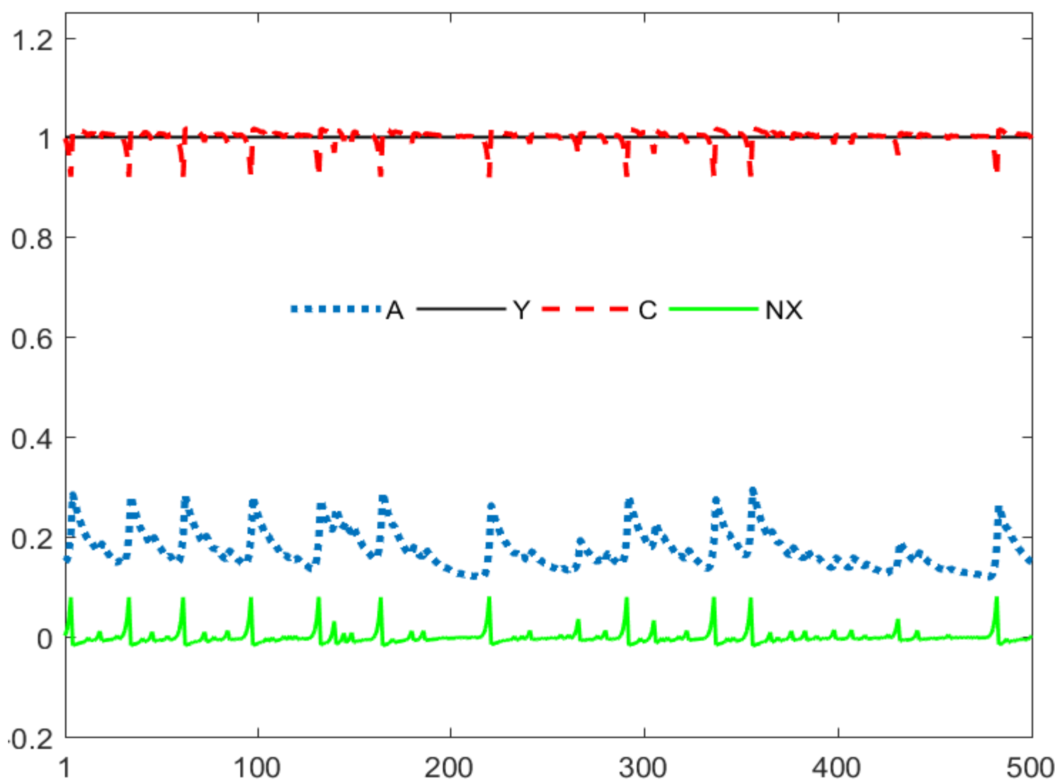


Fig.5 Non-linear RBC model (incomplete capital depreciation, variable labor) with bubbles
Simulated paths of the capital stock (K, blue dotted line), GDP (Y, continuous black line), consumption (C, red dashed line), investment (I, blue dash-dotted line) and hours worked (L, red dotted line) are shown. GDP, C and I series are normalized by steady state GDP. Capital and hours series are normalized by their respective steady states.



(a) Simulated series with constant bust probability: π_t=0.5.



(b) Simulated series with state-contingent bust probability: π_t=0.5 for $A_{t+1}/y \leq 0.25$; $\pi_t \cong 1$ for $A_{t+1}/y > 0.25$

Fig.6 Non-linear Small Open Economy model with bubbles

Simulated paths of net foreign assets (A, dotted blue line), GDP (Y, continuous black line), consumption (C, dashed red line) and net exports (NX, continuous green line). All series are normalized by steady state GDP.