Technical Appendix for

"A Dynamic Equilibrium Model of International Portfolio Holdings: Comment"

Econometrica (2006) Vol. 74, pp.269-273

By Robert Kollmann

Dept. of Economics., University of Paris XII,
F-94010 Créteil Cedex, France
robert_kollmann@yahoo.com
http://www.robertkollmann.com

Center for Economic Policy Research, UK

November 30, 2005

Section A of this Appendix provides further discussions of the error in Serrat's analysis.

Sections B provide more intuition on Serrat's economy. Serrat considers an economy with continuous time, and he uses specialized mathematics. To better understand Serrat's economy, I found it useful to study a discrete time variant of that economy. Solving that variant is straightforward. All key results of the Comment can be replicated in the discrete time structure. In particular, equilibrium portfolios in the discrete time variant have the same structure as portfolios in the (correctly solved) continuous time model.

A. Detailed discussion of error in Serrat paper

In what follows, all page numbers refer to Serrat's paper; **"Comment" refers to my document "A Dynamic Equilibrium Model of International Portfolio Holdings: Comment".**

Serrat shows that equilibrium portfolios have to satisfy the following equation, for countries i=1,2 (see bottom of p.1484):

$$\Lambda_t \pi_t^{i}{}' = \widehat{\Phi}_t^{i}{}' \qquad \text{(A.1)}$$

with $\pi_t^i \equiv (\pi_{1,t}^i, \pi_{2,t}^i, \pi_{3,t}^i, \pi_{4,t}^i)$, where $\pi_{j,t}^i$ is the value of the stock j shares held by country i at date t ($\pi_{j,t}^i \equiv S_{j,t}^i P_{j,t}$); Λ_t is a 4×4 matrix shown on p.1483, and $\widehat{\Phi}_t^i$ is a 1×4 vector shown on p.1484 (see eqn. (4) in Comment).

In what follows, I establish the following facts (used in the Comment):

- Λ_t is singular (Sect. A.1.).
- $\widehat{\Phi}_t^{1}{}'$ equals the third column of Λ_t times $(1+\frac{q}{p})P_{3,t}$ (and $\widehat{\Phi}_t^{2}{}'$ equals the fourth column of Λ_t times $(1+\frac{q}{p})P_{4,t}$); see Sect. A.2.
- When the stock market clears and equation (A.1) holds for one country, then eqn. (A.1) holds for the other country as well (Sect. A.3).

In Sect. A.4, I substitute Serrat's equilibrium portfolio holdings (as described in his Theorem 2) into eqn. (A.1), and confirm (using the facts derived in Sect. A.1-A.3) that those portfolio holdings do **not** satisfy eqn. (A.1).

In Sect. A.5, I use the same approach to verify that the equilibrium stock holdings described by eqn. (5) in the Comment do satisfy eqn. (A.1).

Sect. A.6 provides a detailed discussion of the steps used to derive equilibrium stockholdings (eqn. (5)) in the Comment.

A.1. Λ_t is singular

For notational simplicity, the Comment and the following discussion assume that the utility weight on traded good consumption is identical across countries: $p^1=p^2=p$; Serrat uses the same assumption in his numerical simulations. My key results continue to hold under $p^1 \neq p^2$.

The diffusion matrix of stock prices is $\sigma_t^G = \Lambda_t' \sigma$, where σ is the diffusion matrix of endowments (see p.1483). I carefully checked (and re-derived) the expression for Λ_t shown on p.1483: that expression is correct (the definitions of the variables $u_t^1, u_t^2, \upsilon_t^1, \upsilon_t^2, a_t^1, a_t^2, b_t^1, b_t^2, c_t^1, c_t^2, d_t^1, d_t^2$ that appear in that formula are likewise correct). These checks are available on request.

The Comment shows that $p_{3,t}\delta_{3,t}+p_{4,t}\delta_{4,t}=\frac{p}{q}[\delta_{1,t}+p_{2,t}\delta_{2,t}]$, where $p_{j,t}$ $[\delta_{j,t}]$ is the price [endowment] of good j (with $p_{1,t}\equiv 1$, as good 1 is the numéraire). The price of stock j is: $P_{j,t}=E_t\int_t^T(\xi_s/\xi_t)p_{j,s}\delta_{j,s}ds$ where ξ_s is the equilibrium Arrow-Debreu state price density shown in Serrat's eqn. (11). Thus, stock prices are collinear: $P_{3,t}+P_{4,t}=\frac{p}{q}[P_{1,t}+P_{2,t}]$. Using these facts and the formulae for goods prices and consumptions shown in Serrat's eqns. (12)-(13), it can easily be shown that $u_t^1,u_t^2,\upsilon_t^1,\upsilon_t^2,a_t^1,a_t^2,b_t^1,b_t^2,c_t^1,c_t^2,d_t^1$ and d_t^2 satisfy these restrictions (a derivation is available upon request):

$$a_t^1+b_t^1=1;\ a_t^2+b_t^2=1;\ c_t^1=(1-q)+qd_t^1;\ c_t^2=(1-q)+qd_t^2;\ u_t^1+\upsilon_t^1=1;\ u_t^2+\upsilon_t^2=1 \qquad \text{(A.2)}$$

$$b_t^1P_{3,t}+b_t^2P_{4,t}=\tfrac{p}{q}P_{2,t};\ \ c_t^2P_{4,t}-c_t^1P_{3,t}=(1-q)(P_{4,t}-P_{3,t});\ \ u_t^1P_{1,t}+u_t^2P_{2,t}=\tfrac{q}{p}P_{3,t};\ \ u_t^2P_{2,t}=\tfrac{q}{p}b_t^1P_{3,t}. \qquad \text{(A.3)}$$

Using (A.2) (and setting $p^1=p^2=p$), the matrix Λ_t can be written as:

$$\Lambda_t=\begin{bmatrix} 1 & 1-q & 1-qb_t^1 & 1-qb_t^2 \\ 0 & q & qb_t^1 & qb_t^2 \\ p(u_t^1-\alpha_t) & p(u_t^2-\alpha_t) & \frac{p}{1-q}c_t^1-p\alpha_t & -\frac{p}{1-q}c_t^2+p(1-\alpha_t) \\ -p(u_t^1-\alpha_t) & -p(u_t^2-\alpha_t) & -\frac{p}{1-q}c_t^1+p\alpha_t & \frac{p}{1-q}c_t^2-p(1-\alpha_t) \end{bmatrix}. \qquad \text{(A.4)}$$

The fourth row of Λ_t equals the third row multiplied by -1. Thus, Λ_t is singular.

Let $\widetilde{\Lambda}_t$ be le matrix consisting of the first three rows of Λ_t. The rows of $\widetilde{\Lambda}_t$ are linearly independent. To see this, note that the matrix consisting of the first three columns of $\widetilde{\Lambda}_t$ is non-singular (the determinant of that matrix is $qp\{c_t^1/(1-q)-[b_t^1u_t^2+(1-b_t^1)u_t^1]\}$; this expression differs from zero because $c_t^1/(1-q)>1$ and $0<b_t^1,u_t^1,u_t^2<1$, as can be seen from the definitions of c_t^1,b_t^1,u_t^1,u_t^2 on p.1483).

A.2. Properties of $\widehat{\Phi}_t^i$

I first correct typos in the $\widehat{\Phi}_t^1$ vector reported by Serrat (Sect. A.2.1). I also re-derive $\widehat{\Phi}_t^1$ from scratch (Sect. A.2.3).

A.2.1. Correcting typos

The vector $\widehat{\Phi}_t^1$ is defined in Serrat's eqn. (33). That equation contains several typos:

(i) in the second element of $\widehat{\Phi}_t^1$, the variable c_{2s}^1 has to be multiplied by p_{2s} (and not by ϕ_s: that variable is not defined);

(ii) in the third and fourth elements of $\widehat{\Phi}_t^1$, c_{2s} (not defined) has to be replaced by c_{2s}^1;

(iii) in the fourth element of $\widehat{\Phi}_t^1$, the term $p(1-\alpha_t)X_t$ has to appear with a *negative* sign.

The term X_t that appears in the first, third and fourth elements of the vector $\widehat{\Phi}_t^1$ is defined by Serrat as $X_t \equiv E_t(\int_t^T \frac{\xi_s}{\xi_t}(c_{1s}+p_{2s}c_{2s}+p_{3s}\delta_{3s})ds)$ (see p.1484). In that expression, Serrat forgot the superscript on c_{1s} and c_{2s} (the quantities c_{1s} and c_{2s} are not defined). The correct expression for X_t is thus: $X_t \equiv E_t(\int_t^T \frac{\xi_s}{\xi_t}(c_{1s}^1+p_{2s}c_{2s}^1+p_{3s}\delta_{3s})ds)$, i.e. X_t equals the value at t of country 1's efficient consumption spending process $\{c_{1s}^1+p_{2s}c_{2s}^1+p_{3s}\delta_{3s}\}_{s=t}^T$, evaluated using the Arrow-Debreu pricing kernel $\frac{\xi_s}{\xi_t}$. Note that X_t corresponds to the variable X_t^1 defined on p.1471 (wealth of county 1 at date t): in an efficient equilibrium, country i's wealth at date t, X_t^i, equals the present value of $i's$ efficient consumption expenditures in periods $s \geq t$. In what follows, I thus replace X_t by X_t^1.

It should also be noted that $\widehat{\Phi}_t^1$ is a row vector (in (33), Serrat writes $\widehat{\Phi}_t^1$ as a column vector).

The correct form of Serrat's equation (33) is thus (setting $p^1 = p^2 = p$):

$$\widehat{\Phi}_t^1 = \begin{bmatrix} (1-q)X_t^1 + qE_t(\int_t^T \frac{\xi_s}{\xi_t}(c_{1s}^1 + \varphi_s p_{3s}\delta_{3s})ds) \\ qE_t(\int_t^T \frac{\xi_s}{\xi_t}(p_{2s}c_{2s}^1 + (1-\varphi_s)p_{3s}\delta_{3s})ds) \\ \frac{p}{1-q}E_t(\int_t^T \frac{\xi_s}{\xi_t}(1-q\alpha_s)(c_{1s}^1 + p_{2s}c_{2s}^1 + p_{3s}\delta_{3s})ds) - p\alpha_t X_t^1 \\ -\frac{p}{1-q}qE_t(\int_t^T \frac{\xi_s}{\xi_t}(1-\alpha_s)(c_{1s}^1 + p_{2s}c_{2s}^1 + p_{3s}\delta_{3s}ds)ds) - p(1-\alpha_t)X_t^1 \end{bmatrix}', \qquad \text{(A.5)}$$

with $\varphi_s \equiv (\delta_{1,s})^q / [(\delta_{1,s})^q + (\delta_{2,s})^q]$ (see p.1483).

Explanations of corrections (i)-(iii) listed above:

Corrections (i) and (ii): see the second and third term on the right-hand side of (38). NB: it follows from (37) that the left-hand side of (38) should be ψ_t^1/ξ_t (instead of ψ_t^1).

Correction (iii):
The terms $(1-q)X_t^1$, $p\alpha_t X_t^1$ and $p(1-\alpha_t)X_t$ in the first, third and fourth elements of $\widehat{\Phi}_t^1$ can be traced back to the vector θ_t in Serrat's equations (7) and (34). (NB It follows from (7) that (34) should be written as $\Phi_t^1 \equiv \pi_t^1\sigma_t^G = \psi_t^1/\xi_t + X_t^1\theta_t$.) It follows from the equation in the fifth line of p.1483 that $\theta_t = (1-q)\sigma_1 - p\alpha_t\sigma_3 - p(1-\alpha_t)\sigma_4$, where σ_i is the $i\text{-}th$ row of σ, the diffusion matrix of the vector of (log) endowments (see p.1470). Thus, $\theta_t = H_t\sigma$, with $H_t \equiv ((1-q), 0, -p\alpha_t, -p(1-\alpha_t))$.

Note: it follows from Serrat's eqn. (6) that $d\xi_t/\xi_t = \mu_t^\xi dt - \theta_t dW_t$, for some drift term μ_t^ξ. Applying Itô's Lemma to Serrat's eqn. (11) (that defines ξ_t) confirms that $\theta_t = ((1-q), 0, -p\alpha_t, -p(1-\alpha_t))\sigma$; see Sect. A.7 below.

The corrected eqn. (38) (with ψ_t^1/ξ_t on the left-hand side) can be written as: $\psi_t^1/\xi_t = K_t\sigma$, where $K_t \equiv (K_{1,t}, K_{2,t}, K_{3,t}, K_{4,t})$ is a row vector with:

$$K_{1,t} \equiv qE_t\left(\int_t^T \tfrac{\xi_s}{\xi_t}(c_{1s}^1 + \varphi_s p_{3s}\delta_{3s})ds\right),$$

$$K_{2,t} \equiv qE_t\left(\int_t^T \tfrac{\xi_s}{\xi_t}(p_{2s}c_{2s}^1 + (1-\varphi_s)p_{3s}\delta_{3s})ds\right),$$

$$K_{3,t} \equiv \tfrac{p}{1-q}E_t\left(\int_t^T \tfrac{\xi_s}{\xi_t}(1-q\alpha_s)(c_{1s}^1 + p_{2s}c_{2s}^1 + p_{3s}\delta_{3s})ds\right),$$

$$K_{4,t} \equiv -\tfrac{p}{1-q}qE_t\left(\int_t^T \tfrac{\xi_s}{\xi_t}(1-\alpha_s)(c_{1s}^1 + p_{2s}c_{2s}^1 + p_{3s}\delta_{3s}ds)ds\right). \tag{A.6}$$

As $\Phi_t^1 \equiv \widehat{\Phi}_t^1\sigma$ (see line 1 on p.1484), (34) can be written as: $\widehat{\Phi}_t^1\sigma = K_t\sigma + X_t^1 H_t\sigma$. As σ is assumed to be non-singular (see p.1470), (34) is equivalent to $\widehat{\Phi}_t^1 = K_t + X_t^1 H_t$. Thus,

$$\widehat{\Phi}_t^1 = \left((1-q)X_t^1 + K_{1,t},\ K_{2,t},\ K_{3,t} - p\alpha_t X_t^1,\ K_{4,t} - p(1-\alpha_t)X_t^1\right). \tag{A.7}$$

Substituting (A.6) into (A.7) gives (A.5) (the corrected version of Serrat's eqn. (33)). (A.7) shows that, in the fourth element of $\widehat{\Phi}_t^1$, the term $p(1-\alpha_t)X_t^1$ has to appear with a *negative* sign.

A.2.2. $\widehat{\Phi}_t^{1\prime}$ *equals the third column of* Λ_t *multiplied by* $(1+\frac{q}{p})P_{3,t}$

It follows from Serrat's formulae for consumptions and goods prices in an efficient equilibrium (12), (13) that $c_{1s}^1 = \alpha_s\delta_{1,s}$, $p_{2s}c_{2s}^1 = \alpha_s(\delta_{1,s})^{1-q}(\delta_{2,s})^q$ and $p_{3s}\delta_{3s} = \frac{p}{q}\alpha_s(\delta_{1,s} + (\delta_{1,s})^{1-q}(\delta_{2,s})^q)$. This implies that (in an efficient equilibrium) country 1's consumption spending is proportional to the value of the endowment of country 1's non-traded good:

$$c_{1s}^1 + p_{2s}c_{2s}^1 + p_{3s}\delta_{3s} = (1+\tfrac{q}{p})p_{3,s}\delta_{3,s}. \tag{A.8}$$

Thus, $X_t^1 \equiv E_t\left(\int_t^T \frac{\xi_s}{\xi_t}(c_{1s} + p_{2s}c_{2s} + p_{3s}\delta_{3s})ds\right)$ can be expressed as:

$$X_t^1 = E_t\int_t^T \tfrac{\xi_s}{\xi_t}(1+\tfrac{q}{p})p_{3,s}\delta_{3,s}ds = (1+\tfrac{q}{p})P_{3,t}. \tag{A.9}$$

Hence, in an efficient equilibrium, country 1's wealth at date t equals the price of the "tree" that generates country 1's non-traded good $(P_{3,t})$, multiplied by $(1+\frac{q}{p})$.

A similar reasoning shows that, in an efficient equilibrium, country 2's wealth at date t equals the price of the "tree" that generates country 2's non-traded good $(P_{4,t})$, multiplied by $(1+\frac{q}{p})$:

$$X_t^2 = (1+\tfrac{q}{p})P_{4,t}. \tag{A.10}$$

Simplifying the expression for $\widehat{\Phi}_t^{1\prime}$:

Let $\widehat{\Phi}_{k,t}^1$ be the $k-th$ element of the vector $\widehat{\Phi}_t^1$.

(A.5), (A.9) and Serrat's eqn. (13) imply: $\widehat{\Phi}_{1,t}^1 = (1-q)(1+\frac{q}{p})P_{3,t} + qE_t\left(\int_t^T \frac{\xi_s}{\xi_t}(\alpha_s\delta_{1,s} + \varphi_s p_{3s}\delta_{3s})ds\right)$. It follows from the definitions of u_t^1 and a_t^1 on p.1483 that $u_t^1 P_{1,t} = E_t\int_t^T \frac{\xi_s}{\xi_t}\alpha_s\delta_{1,s}ds$ and $a_t^1 P_{3,t} = E_t\int_t^T \frac{\xi_s}{\xi_t}\varphi_s p_{3s}\delta_{3s}ds$. Thus,

$$\widehat{\Phi}^1_{1,t} = (1-q)(1+\tfrac{q}{p})P_{3,t} + qu^1_t P_{1,t} + qa^1_t P_{3,t}\,. \qquad \text{(A.11)}$$

It follows from (A.2) and (A.3) that $u^1_t P_{1,t} = \frac{q}{p} a^1_t P_{3,t}$. Substituting this into (A.11), gives (using $b^1_t = 1 - a^1_t$):

$$\widehat{\Phi}^1_{1,t} = (1-qb^1_t)(1+\tfrac{q}{p})P_{3.t}\,.$$

Similarly, one can show that

$$\widehat{\Phi}^1_{2,t} = qu^2_t P_{2,t} + qb^1_t P_{3,t} = qb^1_t(1+\tfrac{q}{p})P_{3,t}\,,$$

$$\widehat{\Phi}^1_{3,t} = [\tfrac{p}{1-q}c^1_t - p\alpha_t](1+\tfrac{q}{p})P_{3,t,}$$

$$\widehat{\Phi}^1_{4,t} = [-\tfrac{p}{1-q}c^1_t + p\alpha_t](1+\tfrac{q}{p})P_{3,t.}$$

Note that $\widehat{\Phi}^1_{4,t} = -\widehat{\Phi}^1_{3,t}$.

Thus:
$$\widehat{\Phi}^{1\,\prime}_t = \begin{bmatrix} 1-qb^1_t \\ qb^1_t \\ \frac{p}{1-q}c^1_t - p\alpha_t \\ -\frac{p}{1-q}c^1_t + p\alpha_t \end{bmatrix}(1+\tfrac{q}{p})P_{3.t} \qquad \text{(A.12)}$$

We see from (A.4) and (A.12) that $\widehat{\Phi}^{1\,\prime}_t$ equals the third column of Λ_t multiplied by $(1+\frac{q}{p})P_{3.t}$.

For $i=1$, one can thus write (A.1) as: $\Lambda_t(\pi^1_{1,t}, \pi^1_{2,t}, \pi^1_{3,t} - (1+\frac{q}{p})P_{3,t}, \pi^1_{4,t})' = 0_4$, where 0_k is a $k\times 1$ vector of zeros. Recall that $\pi^1_{j,t} \equiv P_{j,t}S^1_{j,t}$, and that the fourth row of Λ_t is proportional to the third row. Thus, (A.1) holds for $i=1$ if and only if equation (6) in the Comment holds:

$$\widehat{\Lambda}_t\,(P_{1,t}S^1_{1,t}, P_{2,t}S^1_{2,t}, P_{3,t}(S^1_{3,t} - (1+\tfrac{q}{p})), P_{4,t}S^1_{4,t})' = 0_3,$$

where $\widehat{\Lambda}_t$ is the matrix consisting of the first 3 rows of Λ_t.

A.2.3. Alternative derivation of the fact that $\widehat{\Phi}^{1\,\prime}_t$ equals the third column of Λ_t multiplied by $(1+\frac{q}{p})P_{3,t}$

From Serrat's equation (7) and his Definition 1 (p.1472): $\Phi^i_t = \frac{1}{\xi_t}\psi^i_t + X^i_t\theta_t$, where ψ^i_t is the process defined by $\int_0^t \psi^i_s dW_s = \zeta^i_t \equiv E_t(\int_0^T \xi_s C^i_s p_s ds) - E(\int_0^T \xi_s C^i_s p_s ds)$, where W_s is the Wiener process that governs the (log) endowments, and $C^i_s p_s$ represents (in Serrat's notation) country $i's$ consumption spending at date s ($C^1_s p_s \equiv c^1_{1,s} + p_{2,s}c^1_{2,s} + p_{3,s}\delta_{3,s}$, $C^2_s p_s \equiv c^2_{1,s} + p_{2,s}c^2_{2,s} + p_{4,s}\delta_{4,s}$). (Notice typo in Serrat's definition of ζ^i_t in his eqn. (8); correct definition can be found after eqn. (34) on p.1484.)

Note that $\zeta^i_t = \int_0^t \xi_s C^i_s p_s ds + E_t(\int_t^T \xi_s C^i_s p_s ds) - E(\int_0^T \xi_s C^i_s p_s ds) = \int_0^t \xi_s C^i_s p_s ds + X^i_t \xi_t - E(\int_0^T \xi_s C^i_s p_s ds))$, where I used $X^i_t = E_t(\int_t^T (\xi_s/\xi_t) C^i_s p_s ds)$ (see Sect. A.2.1). Thus:

$$d\zeta_t^i = \xi_t C_t^i p_t dt + d(X_t^i \xi_t). \quad \text{(A.13)}$$

It follows from the discussion in Sect. A.2.2. that $C_t^1 p_t = (1+\frac{q}{p}) p_{3,t}\delta_{3,t}$, $C_t^2 p_t = (1+\frac{q}{p}) p_{4,t}\delta_{4,t}$, $X_t^1 = (1+\frac{q}{p}) P_{3,t}$, $X_t^2 = (1+\frac{q}{p}) P_{4,t}$. Note that $d(X_t^i \xi_t) = \mu_t^{X^i\xi} dt + dX_t^i \xi_t + X_t^i \xi_t (d\xi_t/\xi_t)$, for some term $\mu_t^{X\xi}$. Substituting this into (A.13) gives

$$\frac{1}{\xi_t} d\zeta_t^1 = (\mu_t^{X^1\xi}/\xi_t)dt + (1+\tfrac{q}{p}) p_{3,t}\delta_{3,t} dt + (1+\tfrac{q}{p}) dP_{3,t} + X_t^1 (d\xi_t/\xi_t),$$

$$\frac{1}{\xi_t} d\zeta_t^2 = (\mu_t^{X^2\xi}/\xi_t)dt + (1+\tfrac{q}{p}) p_{4,t}\delta_{4,t} dt + (1+\tfrac{q}{p}) dP_{4,t} + X_t^2 (d\xi_t/\xi_t).$$

It follows from Serrat's eqn. (2) that

$$p_{3,t}\delta_{3,t} dt + dP_{3,t} = P_{3,t}(\mu_{3,t}^G dt + \sigma_{3,t}^G dW_t) \quad \text{and} \quad p_{4,t}\delta_{4,t} dt + dP_{4,t} = P_{4,t}(\mu_{4,t}^G dt + \sigma_{4,t}^G dW_t),$$

where $\mu_{k,t}^G$ and $\sigma_{k,t}^G$ are the $k-th$ element of the μ_t^G and the $k-th$ row of σ_t^G, respectively.

As shown in Sect. A.7, $d\xi_t/\xi_t = \mu_t^\xi dt - \theta_t dW_t$, with $\theta_t = ((1-q), 0, -p\alpha_t, -p(1-\alpha_t))\sigma$. Hence,

$$\frac{1}{\xi_t} d\zeta_t^1 = (\mu_t^{X^1\xi}/\xi_t)dt + (1+\tfrac{q}{p}) P_{3,t}(\mu_{3,t}^G dt + \sigma_{3,t}^G dW_t) + X_t^1(\mu_t^\xi dt - \theta_t dW_t) \quad \text{and}$$

$$\frac{1}{\xi_t} d\zeta_t^2 = (\mu_t^{X^2\xi}/\xi_t)dt + (1+\tfrac{q}{p}) P_{4,t}(\mu_{4,t}^G dt + \sigma_{4,t}^G dW_t) + X_t^2(\mu_t^\xi dt - \theta_t dW_t).$$

Note that $E_t d\zeta_t^i = 0$. ($d\zeta_t^i = \int_t^{t+dt} \xi_s C_s^i p_s ds + E_{t+dt}\int_{t+dt}^T \xi_s C_s^i p_s ds - E_t \int_t^T \xi_s C_s^i p_s ds$; thus: $E_t d\zeta_t^i = E_t \int_t^{t+dt} \xi_s C_s^i p_s ds + E_t \int_{t+dt}^T \xi_s C_s^i p_s ds - E_t \int_t^T \xi_s C_s^i p_s ds = E_t \int_t^T \xi_s C_s^i p_s ds - E_t \int_t^T \xi_s C_s^i p_s ds = 0.$)

This implies that the drift terms $\mu_t^{X^1\xi}, \mu_t^{X^2\xi}, \mu_t^G$ and μ_t^ξ have to satisfy the following restrictions: $\mu_t^{X^1\xi}/\xi_t + (1+\frac{q}{p}) P_{3,t}\mu_{3,t}^G + X_t^1 \mu_t^\xi = 0$ and $\mu_t^{X^2\xi}/\xi_t + (1+\frac{q}{p}) P_{4,t}\mu_{4,t}^G + X_t^2 \mu_t^\xi = 0$. Hence,

$$\frac{1}{\xi_t} d\zeta_t^1 = \{(1+\tfrac{q}{p}) P_{3,t}\sigma_{3,t}^G - X_t^1\theta_t\} dW_t \quad \text{and} \quad \frac{1}{\xi_t} d\zeta_t^2 = \{(1+\tfrac{q}{p}) P_{4,t}\sigma_{4,t}^G - X_t^2\theta_t\} dW_t.$$

Thus, the process ψ_t^i defined by $\zeta_t^i = \int_0^t \psi_s^i dW_s$ is given by:

$$\psi_t^1 = \xi_t\{(1+\tfrac{q}{p}) P_{3,t}\sigma_{3,t}^G - X_t^1\theta_t\} \quad \text{and} \quad \psi_t^3 = \xi_t\{(1+\tfrac{q}{p}) P_{4,t}\sigma_{4,t}^G - \theta_t\}.$$

This implies that $\Phi_t^i = \dfrac{\psi_t^i}{\xi_t} + X_t^i\theta_t$ (see Serrat's eqn. (7) and Definition 1 on p.1472) can be expressed as

$$\Phi_t^1 = (1+\tfrac{q}{p}) P_{3,t}\sigma_{3,t}^G \quad \text{and} \quad \Phi_t^2 = (1+\tfrac{q}{p}) P_{4,t}\sigma_{4,t}^G.$$

Note that $\sigma_t^G = \Lambda_t{}'\sigma$ (see p.1483). Thus, $\sigma_{k,t}^G = \lambda_{k,t}{}'\sigma$, where $\lambda_{k,t}$ is the $k-th$ column of Λ_t. Serrat defines the vector $\widehat{\Phi}_t^i$ by: $\Phi_t^i = \widehat{\Phi}_t^i \sigma$ (see first line on p.1484). As σ is assumed to be non-singular (p.1470), it follows from the preceding expressions that

$$\widehat{\Phi}_t^{1\prime} = \lambda_{3,t}(1+\tfrac{q}{p}) P_{3,t} \quad \text{and} \quad \widehat{\Phi}_t^{2\prime} = \lambda_{4,t}(1+\tfrac{q}{p}) P_{4,t}.$$

This confirms that $\widehat{\Phi}_t^{1\prime}$ equals the third column of Λ_t multiplied by $(1+\frac{q}{p}) P_{3,t}$; in addition, I have shown that $\widehat{\Phi}_t^{2\prime}$ equals the fourth column of Λ_t multiplied by $(1+\frac{q}{p}) P_{4,t}$.

A.3. When the stock market clears and eqn. (A.1) holds for one country, then (A.1) holds for the other country as well.

Stock market clearing requires: $S^1_{j,t}+S^2_{j,t}=1$ for $j=1,..,4$, or equivalently $\pi^1_{j,t}+\pi^2_{j,t}=P_{j,t}$. Assume that this condition is met, and that eqn. (A.1) holds for $i=1$. I now show that then (A.1) holds for $i=2$ as well: $\pi^2_t=P_t-\pi^1_t$, with $P_t\equiv(P_{1,t},P_{2,t},P_{3,t},P_{4,t})$ solves $\Lambda_t\pi^{2}_t{}'=\widehat{\Phi}^2_t{}'$.

Note that $\Lambda_t(P_t-\pi^1_t)'=\widehat{\Phi}^2_t{}'$ $\Leftrightarrow$ $\Lambda_t P_t'=\widehat{\Phi}^1_t{}'+\widehat{\Phi}^2_t{}'$ (if, as assumed $\Lambda_t\pi^1_t{}'=\widehat{\Phi}^1_t{}'$). As $\widehat{\Phi}^1_t{}'$ [$\widehat{\Phi}^2_t{}'$] equals the third [fourth] column of Λ_t multiplied by $(1+\frac{q}{p})P_{3,t}$ [$(1+\frac{q}{p})P_{4,t}$], this equation can be written as $\Lambda_t Q_1 P_t'=0$, where $Q_1\equiv\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-q/p&0\\0&0&0&-q/p\end{bmatrix}$. Premultiplying $\Lambda_t Q_1 P_t'=0$ by the non-singular matrix $Q_2\equiv\begin{bmatrix}1&1&0&0\\0&1/q&0&0\\\alpha_t&\alpha_t&1/p&0\\-\alpha_t&-\alpha_t&0&1/p\end{bmatrix}$ gives:

$$\begin{bmatrix}1&1&-q/p&-q/p\\0&1&-(q/p)b^1_t&-(q/p)b^2_t\\u^1_t&u^2_t&-(q/p)c^1_t/(1-q)&(q/p)(c^2_t/(1-q)-1)\\-u^1_t&-u^2_t&(q/p)c^1_t/(1-q)&-(q/p)(c^2_t/(1-q)-1)\end{bmatrix}P_t'=0_4.$$

Using $P_{3,t}+P_{4,t}=\frac{p}{q}[P_{1,t}+P_{2,t}]$ and (A.3) it can be shown that this statement is true (which proves that $\Lambda_t(P_t-\pi^1_t)'=\widehat{\Phi}^2_t{}'$ when $\Lambda_t\pi^1_t{}'=\widehat{\Phi}^1_t{}'$):

i) First row:
$P_{1,t}+P_{2,t}-(q/p)P_{3,t}-(q/p)P_{4,t}=0$ holds because $P_{3,t}+P_{4,t}=\frac{p}{q}[P_{1,t}+P_{2,t}]$.
ii) Second row:
$P_{2,t}-(q/p)b^1_tP_{3,t}-(q/p)b^2_tP_{4,t}=0$ holds because $b^1_tP_{3,t}+b^2_tP_{4,t}=\frac{p}{q}P_{2,t}$.
ii) Third and fourth rows:
$u^1_tP_{1,t}+u^2_tP_{2,t}-P_{3,t}(q/p)c^1_t/(1-q)+P_{4,t}(q/p)(c^2_t/(1-q)-1)=0$ holds because $u^1_tP_{1,t}+u^2_tP_{2,t}=\frac{q}{p}P_{3,t}$ and $c^2_tP_{4,t}-c^1_tP_{3,t}=(1-q)(P_{4,t}-P_{3,t})$.

A.4. Additional proof that Serrat's equilibrium portfolio holdings do not satisfy eqn. (A.1)

The Comment shows that (A.1) implies a restriction on stockholdings (see eqn. (5) in Comment) that Serrat's equilibrium portfolio holdings (see eqn. (17) in his Theorem 2) do not meet.

The fact that Serrat's portfolio holdings are inconsistent with equilibrium can also be shown directly by substituting Serrat's portfolio holdings into (A.1): those portfolios do not satisfy $\Lambda_t \pi_t^{i}{}' = \widehat{\Phi}_t^{i}{}'$. The portfolio holdings in Serrat's equation (17) can be written as: $\pi_t^1 = (u_t^1 P_{1,t}, u_t^2 P_{2,t}, P_{3,t}, 0)$, $\pi_t^2 = ((1-u_t^1)P_{1,t}, (1-u_t^2)P_{2,t}, 0, P_{4,t})$, where u_t^1 and u_t^2 are defined on p.1483.

Let $\Lambda_{k,t}$ and $\widehat{\Phi}_{k,t}^{i}$ be the $k-th$ row of Λ_t and the $k-th$ element of $\widehat{\Phi}_{k,t}^{i}$, respectively.

Using (A.3), one can show that $\Lambda_{\mathrm{k,t}}(u_t^1 P_{1,t}, u_t^2 P_{2,t}, P_{3,t}, 0)' \neq \widehat{\Phi}_{k,t}^{1}$ for $k=3$ and for $k=4$, which establishes that the portfolios shown in Serrat's Theorem 2 are inconsistent with equilibrium.

Proof: If $\Lambda_{3,\mathrm{t}}(u_t^1 P_{1,t}, u_t^2 P_{2,t}, P_{3,t}, 0)' = \widehat{\Phi}_{3,t}^{1}$ were true, then one would have that

$$p(u_t^1 - \alpha_t)u_t^1 P_{1,t} + p(u_t^2 - \alpha_t)u_t^2 P_{2,t} + [\tfrac{1}{1-q} p c_t^1 - p\alpha_t] P_{3,t} = (1+\tfrac{q}{p})[\tfrac{1}{1-q} p c_t^1 - p\alpha_t] P_{3,t} \Leftrightarrow$$

$$(u_t^1 - \alpha_t)u_t^1 P_{1,t} + (u_t^2 - \alpha_t)u_t^2 P_{2,t} = \tfrac{q}{p}[\tfrac{1}{1-q} c_t^1 - \alpha_t] P_{3,t} \Leftrightarrow$$

$$(u_t^1)^2 P_{1,t} + (u_t^2)^2 P_{2,t} = \tfrac{q}{p}\tfrac{1}{1-q} c_t^1 P_{3,t}, \qquad \text{(A.14)}$$

where I used the fact that $u_t^1 P_{1,t} + u_t^2 P_{2,t} = \frac{q}{p} P_{3,t}$ (see (A.3)). It follows from the definition of c_t^1 on p.1483 that $\frac{1}{1-q} c_t^1 > 1$ and that $0 < u_t^1, u_t^2 < 1$. Thus, the right-hand side of (A.14) is greater than $\frac{q}{p} P_{3,t}$ ($\frac{q}{p}\frac{1}{1-q} c_t^1 P_{3,t} > \frac{q}{p} P_{3,t}$), while the left-hand side is smaller than $u_t^1 P_{1,t} + u_t^2 P_{2,t}$, and hence smaller than $\frac{q}{p} P_{3,t}$. This implies that (A.14) is false, which proves that $\Lambda_{3,\mathrm{t}}(u_t^1 P_{1,t}, u_t^2 P_{2,t}, P_{3,t}, 0)' \neq \widehat{\Phi}_{3,t}^{1}$.

Note: Serrat's portfolios do satisfy his matrix eqn. (39), which represents the condition $\pi_t^1 + \pi_t^2 = (P_{1,t}, P_{2,t}, P_{3,t}, P_{4,t})$, and a <u>subset</u> of the conditions included in (A.1) (namely: $\Lambda_{\mathrm{k,t}} \pi_t^i = \widehat{\Phi}_{k,t}^{i}$ for $k=1,2$ and $i=1,2$). As only a subset of the equilibrium conditions is used, this is not enough for equilibrium. Also, the 8×8 matrix on the left-hand side of (39) is singular (post-multiplying that matrix by the vectors $(1-b_t^2, b_t^2, 0, -1, -1+b_t^2, -b_t^2, 0, 1)'$ and $(1-b_t^1, b_t^1, -1, 0, b_t^1-1, -b_t^1, 1, 0)'$ yields vectors of zeros, i.e. the columns of the matrix are not linearly independent.). Thus, Serrat's claim that "...the system (39) is exactly identified for the vector (π_t^1, π_t^2)" (p.1485) is incorrect.

A.5. Verifying that the equilibrium stockholdings derived in the Comment satisfy (A.1)

The Comment proves that **any** stock holdings consistent with the following conditions solve eqn. (A.1) for $i=1,2$:

$$S^1_{2,t}=S^1_{1,t},\quad S^1_{3,t}=1+\tfrac{q}{p}-\tfrac{q}{p}S^1_{1,t},\quad S^1_{4,t}=-\tfrac{q}{p}S^1_{1,t};\ \text{ and }\ S^2_{j,t}=1-S^1_{j,t}\ \text{ for } j=1,..,4. \qquad (5)$$

Here, I verify this, by substituting (5) into $\Lambda_t\pi^{1\prime}_t=\widehat{\Phi}^{1\prime}_t$. (As shown above, when the stock market clears, and $\Lambda_t\pi^{1\prime}_t=\widehat{\Phi}^{1\prime}_t$ holds, then $\Lambda_t\pi^{2\prime}_t=\widehat{\Phi}^{2\prime}_t$ holds too.)

Note that (5) implies that $\pi^1_t=(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t}),-P_{4,t}\tfrac{q}{p}S_{1,t})$.

I show that $\Lambda_{k,t}(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t}),-P_{4,t}\tfrac{q}{p}S_{1,t})'=\widehat{\Phi}^1_{k,t}$ holds for k=1,..,4. (Recall that $\Lambda_{k,t}$ $[\widehat{\Phi}^1_{k,t}]$ is the k*–th* row of Λ_t [element of $\widehat{\Phi}^1_{k,t}$].) In what follows, (A.3) is used repeatedly.

***k*=1:**

$$\Lambda_{1,t}(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t}),-P_{4,t}\tfrac{q}{p}S_{1,t})'=\widehat{\Phi}^1_{1,t}\quad\Leftrightarrow$$
$$P_{1,t}S_{1,t}+(1-q)P_{2,t}S_{1,t}+(1-qb^1_t)P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t})-(1-qb^2_t)P_{4,t}\tfrac{q}{p}S_{1,t}=(1-qb^1_t)(1+\tfrac{q}{p})P_{3,t}\ \Leftrightarrow$$
$$P_{1,t}S_{1,t}+(1-q)P_{2,t}S_{1,t}=(1-qb^1_t)P_{3,t}\tfrac{q}{p}S_{1,t}+(1-qb^2_t)P_{4,t}\tfrac{q}{p}S_{1,t}\quad\Leftrightarrow$$
$P_{2,t}S_{1,t}=b^1_tP_{3,t}\tfrac{q}{p}S_{1,t}+b^2_tP_{4,t}\tfrac{q}{p}S_{1,t}$; this equation holds as $\tfrac{p}{q}P_{2,t}=b^1_tP_{3,t}+b^2_tP_{4,t}$.

(NB Above I used the fact that $P_{1,t}+P_{2,t}=\tfrac{q}{p}(P_{3,t}+P_{4,t})$.)

***k*=2:**

$$\Lambda_{2,t}(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t}),-P_{4,t}\tfrac{q}{p}S_{1,t})'=\widehat{\Phi}^1_{2,t}\quad\Leftrightarrow$$
$$qP_{2,t}S_{1,t}+qb^1_tP_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t})-qb^2_tP_{4,t}\tfrac{q}{p}S_{1,t}=qb^1_t(1+\tfrac{q}{p})P_{3,t}\ \Leftrightarrow$$
$P_{2,t}S_{1,t}=b^1_tP_{3,t}\tfrac{q}{p}S_{1,t}+b^2_tP_{4,t}\tfrac{q}{p}S_{1,t}$; this is a true statement as $\tfrac{p}{q}P_{2,t}=b^1_tP_{3,t}+b^2_tP_{4,t}$.

***k*=3:**

$$\Lambda_{3,t}(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t}),-P_{4,t}\tfrac{q}{p}S_{1,t})'=\widehat{\Phi}^1_{3,t}\quad\Leftrightarrow$$
$$p(u^1_t-\alpha_t)P_{1,t}S_{1,t}+p(u^2_t-\alpha_t)P_{2,t}S_{1,t}+[\tfrac{1}{1-q}pc^1_t-p\alpha_t]P_{3,t}(1+\tfrac{q}{p}-\tfrac{q}{p}S_{1,t})+[\tfrac{1}{1-q}pc^2_t-p(1-\alpha)]P_{4,t}\tfrac{q}{p}S_{1,t}=$$
$$[\tfrac{1}{1-q}pc^1_t-p\alpha_t](1+\tfrac{q}{p})P_{3,t}\quad\Leftrightarrow$$

$$(u^1_t-\alpha_t)P_{1,t}S_{1,t}+(u^2_t-\alpha_t)P_{2,t}S_{1,t}-[\tfrac{1}{1-q}c^1_t-\alpha_t]P_{3,t}\tfrac{q}{p}S_{1,t}+[\tfrac{1}{1-q}c^2_t-(1-\alpha)]P_{4,t}\tfrac{q}{p}S_{1,t}=0\quad\Leftrightarrow$$

$$u^1_tP_{1,t}S_{1,t}+u^2_tP_{2,t}S_{1,t}-\tfrac{1}{1-q}c^1_tP_{3,t}\tfrac{q}{p}S_{1,t}+[\tfrac{1}{1-q}c^2_t-1]P_{4,t}\tfrac{q}{p}S_{1,t}=0\quad(\text{as } P_{1,t}+P_{2,t}=\tfrac{q}{p}(P_{3,t}+P_{4,t}))\quad\Leftrightarrow$$

$\tfrac{q}{p}(P_{3,t}-P_{4,t})S_{1,t}+\tfrac{1}{1-q}\tfrac{q}{p}(c^2_tP_{4,t}-c^1_tP_{3,t})S_{1,t}=0$ (as $u^1_tP_{1,t}+u^2_tP_{2,t}=\tfrac{q}{p}P_{3,t}$). This is a true statement as $c^2_tP_{4,t}-c^1_tP_{3,t}=(1-q)(P_{4,t}-P_{3,t})$.

***k*=4:**

Note that $\Lambda_{4,t}=-\Lambda_{3,t}$ and $\widehat{\Phi}^{1}_{4,t}=-\widehat{\Phi}^{1}_{3,t}$. Thus, $\Lambda_{4,t}(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\frac{q}{p}-\frac{q}{p}S_{1,t}),-P_{4,t}\frac{q}{p}S_{1,t})'=\widehat{\Phi}^{1}_{4,t}$ holds, because $\Lambda_{3,t}(P_{1,t}S_{1,t},P_{2,t}S_{1,t},P_{3,t}(1+\frac{q}{p}-\frac{q}{p}S_{1,t}),-P_{4,t}\frac{q}{p}S_{1,t})'=\widehat{\Phi}^{1}_{3,t}$.

A.6. More details on derivation of equilibrium stock holdings

This Section provides additional details on the derivation of equilibrium stock holdings in the Comment.

As $\widehat{\Phi}^{1\prime}_{t}$ equals the third column of Λ_t multiplied by $(1+\frac{q}{p})P_{3,t}$, equation (A.1) for i=1 can be written as:

$$\Lambda_t(\pi^1_{1,t},\pi^1_{2,t},\pi^1_{3,t}-(1+\tfrac{q}{p})P_{3,t},\pi^1_{4,t})'=0_4,\ \text{ with } 0_4\equiv(0,0,0,0)'. \qquad \text{(A.15)}$$

Note that $\pi^1_{j,t}\equiv S^1_{j,t}P_{j,t}$. As can be seen from (A.4), the fourth row of the matrix Λ_t equals its third row multiplied by -1. Thus, (A.15) holds iff eqn. (6) of the Comment holds:

$$\widehat{\Lambda}_t\,(P_{1,t}S^1_{1,t},P_{2,t}S^1_{2,t},P_{3,t}(S^1_{3,t}-(1+\tfrac{q}{p})),P_{4,t}S^1_{4,t})'=0_3,\quad 0_3\equiv(0,0,0)', \qquad (6)$$

where $\widehat{\Lambda}_t$ is the matrix consisting of the first three rows of Λ_t.

Premultiplying (6) by the non-singular matrix $\begin{bmatrix}1 & 1-1/q & 0\\ 0 & 1/q & 0\\ \alpha_t-u^1_t & K_{1,t} & 1/p\end{bmatrix}$ gives eqn. (7) in the Comment:

$$\begin{bmatrix}1 & 0 & 1-b^1_t & 1-b^2_t\\ 0 & 1 & b^1_t & b^2_t\\ 0 & 0 & K_{2,t} & K_{3,t}\end{bmatrix}(P_{1,t}S^1_{1,t},P_{2,t}S^1_{2,t},P_{3,t}(S^1_{3,t}-(1+\tfrac{q}{p})),P_{4,t}S^1_{4,t})'=0_3, \qquad (7)$$

where $u^1_t,u^2_t,b^1_t,b^2_t,c^1_t,c^2_t$ are defined on p.1483 and $K_{1,t}\equiv\alpha_t-u^1_t+[u^1_t-u^2_t]\frac{1}{q}$, $K_{2,t}\equiv(u^1_t-u^2_t)b^1_t+c^1_t\frac{1}{1-q}-u^1_t$, $K_{3,t}\equiv(u^1_t-u^2_t)b^2_t+1-c^2_t\frac{1}{1-q}-u^1_t$, with $K_{2,t}P_{3,t}=-K_{3,t}P_{4,t}\neq 0$.

Proof that $K_{2,t}P_{3,t}=-K_{3,t}P_{4,t}\neq 0$: Substituting the definitions of $K_{2,t}$ and $K_{3,t}$ into $K_{2,t}P_{3,t}=-K_{3,t}P_{4,t}$ gives: $\{(u^1_t-u^2_t)b^1_t+c^1_t\frac{1}{1-q}-u^1_t\}P_{3,t}=-\{(u^1_t-u^2_t)b^2_t+1-c^2_t\frac{1}{1-q}-u^1_t\}P_{4,t}\quad\Leftrightarrow$

$$(u^1_t-u^2_t)(b^1_tP_{3,t}+b^2_tP_{4,t})+(c^1_tP_{3,t}-c^2_tP_{4,t})\tfrac{1}{1-q}=-P_{4,t}+u^1_t(P_{3,t}+P_{4,t}). \qquad \text{(A.16)}$$

As $b^1_tP_{3,t}+b^2_tP_{4,t}=\frac{p}{q}P_{2,t}$, $c^2_tP_{4,t}-c^1_tP_{3,t}=(1-q)(P_{4,t}-P_{3,t})$ (see (A.3)) and $P_{3,t}+P_{4,t}=\frac{p}{q}[P_{1,t}+P_{2,t}]$ (see Sect. (A.1)) eqn. (A.16) can be expressed as:

$$(u^1_t-u^2_t)\tfrac{p}{q}P_{2,t}+P_{3,t}-P_{4,t}=-P_{4,t}+u^1_t\tfrac{p}{q}(P_{1,t}+P_{2,t})\quad\Leftrightarrow$$

$$P_{3,t}=u^1_t\tfrac{p}{q}P_{2,t}+u^2_t\tfrac{p}{q}P_{2,t}\,.$$

This is a true statement (see (A.3)), which proves that $K_{2,t}P_{3,t}=-K_{3,t}P_{4,t}$.

I next show that $K_{2,t}P_{3,t}\neq 0$. As *log* endowments follow a diffusion process, endowments are strictly positive, which ensures that stock prices are also strictly positive. Note that $K_{2,t}=c^1_t\frac{1}{1-q}-[b^1_tu^2_t+(1-b^1_t)u^1_t]$. This expression differs from zero, because $c^1_t/(1-q)>1$ and $0<b^1_t,u^1_t,u^2_t<1$ (which follows from the definitions of c^1_t,b^1_t,u^1_t,u^2_t on p.1483).

Next, I solve eqn. (7) (of Comment) for the country 1 stock holdings. (7) holds iff these three conditions are satisfied:

(i) $$P_{1,t}S^1_{1,t} + (1-b^1_t)P_{3,t}(S^1_{3,t}-(1+\tfrac{q}{p})) + (1-b^2_t)P_{4,t}S^1_{4,t} = 0,$$

(ii) $$P_{2,t}S^1_{2,t} + b^1_t P_{3,t}(S^1_{3,t}-(1+\tfrac{q}{p})) + b^2_t P_{4,t}S^1_{4,t} = 0 \text{ and}$$

(iii) $$K_{2,t}P_{3,t}(S^1_{3,t}-(1+\tfrac{q}{p})) + K_{3,t}P_{4,t}S^1_{4,t} = 0.$$

As $K_{2,t}P_{3,t}=-K_{3,t}P_{4,t} \neq 0$, condition (iii) holds iff $S^1_{3,t}-(1+\tfrac{q}{p})=S^1_{4,t}$. Substituting this into (ii) and using the fact that $b^1_t P_{3,t}+b^2_t P_{4,t}=\tfrac{p}{q}P_{2,t}$ (see (A.3)) gives: $P_{2,t}S^1_{2,t}+\tfrac{p}{q}P_{2,t}S^1_{4,t}=0$; this condition is satisfied iff $S^1_{2,t}=-\tfrac{p}{q}S^1_{4,t}$. Substituting $S^1_{3,t}-(1+\tfrac{q}{p})=S^1_{4,t}$ into (i), and using $b^1_t P_{3,t}+b^2_t P_{4,t}=\tfrac{p}{q}P_{2,t}$ and $P_{3,t}+P_{4,t}=\tfrac{p}{q}(P_{1,t}+P_{2,t})$ gives: $P_{1,t}S^1_{1,t}+\tfrac{p}{q}P_{1,t}S^1_{4,t}=0$; this condition is satisfied iff $S^1_{1,t}=-\tfrac{p}{q}S^1_{4,t}$.

In summary: I have shown that (7) (and thus (6) and (A.1), for $i=1$) hold iff the country 1 stock holdings satisfy these restrictions: $S^1_{3,t}-(1+\tfrac{q}{p})=S^1_{4,t}$, $S^1_{2,t}=-\tfrac{p}{q}S^1_{4,t}$, $S^1_{1,t}=-\tfrac{p}{q}S^1_{4,t}$. These restrictions are equivalent to those listed in (5) of the Comment.

It follows from the discussion in Sect. A.3 that if country 1 stock holdings satisfy (A.1), for $i=1$, then country 2 stock holdings given by $S^2_{j,t}=1-S^1_{j,t}$ ($j=1,..,4$) satisfy (A.1) for $i=2$.

Thus, (A.1) holds for $i=1,2$ iff

$$S^1_{2,t} = S^1_{1,t},\quad S^1_{3,t} = 1+\tfrac{q}{p}-\tfrac{q}{p}S^1_{1,t},\quad S^1_{4,t} = -\tfrac{q}{p}S^1_{1,t};\ \text{ and }\ S^2_{j,t}=1-S^1_{j,t} \text{ for } j=1,..,4. \tag{5}$$

A.7. Deriving θ_t using Itô's Lemma

ξ_t (Arrow-Debreu state price density) is defined in Serrat's equation (11) and can be written as (using $p^1=p^2=p$):

$$\xi_t = (\delta_{1,t})^{q-1}[(\lambda_1)^{1/(q-1)}(\delta_{3,t})^{p/(1-q)} + (\lambda_2)^{1/(q-1)}(\delta_{4,t})^{p/(1-q)}]^{1-q}e^{-\rho t}, \tag{A.17}$$

where λ_1 and λ_2 are date- and state invariant terms; ρ is the subjective discount rate. (For simplicity I assume that both countries have identical subjective discount rates: $\rho_1=\rho_2=\rho$. My results go through when $\rho_1\neq\rho_2$; Serrat's theoretical analysis allows for $\rho_1\neq\rho_2$, but he uses $\rho_1=\rho_2$ in his numerical simulations.)

(A.17) can be written as:

$$\xi_t = (\exp(\ln(\delta_{1,t})))^{q-1}[\lambda_1^{1/(q-1)}(\exp(\ln(\delta_{3,t})))^{p/(1-q)}+(\lambda_2)^{1/(q-1)}\exp(\ln((\delta_{4,t})))^{p/(1-q)}]^{1-q}e^{-\rho t}.$$

Serrat assumes that $e_t \equiv (\ln(\delta_{1,t}),\ln(\delta_{2,t}),\ln(\delta_{3,t}),\ln(\delta_{4,t}))'$ follows a diffusion process:

$$de_t = \mu\, dt + \sigma dW_t,$$

where W_t is a four-dimensional Wiener process (see p.1470).

Itô's lemma implies:

$$d\xi_t = \hat{\mu}_{\xi,t}\, dt + \big(\partial\xi_t/\partial\ln(\delta_{1,t}), \partial\xi_t/\partial\ln(\delta_{2,t}), \partial\xi_t/\partial\ln(\delta_{3,t}), \partial\xi_t/\partial\ln(\delta_{4,t})\big)\, \sigma\, dW_t,$$

for some drift term $\hat{\mu}_{\xi,t}$ (see, e.g., Miranda and Fackler (2002), p.470). Therefore,

$$d\xi_t/\xi_t = \mu_{\xi,t}\, dt + \big(\varepsilon_{\xi_t,\delta_{1,t}}, \varepsilon_{\xi_t,\delta_{2,t}}, \varepsilon_{\xi_t,\delta_{3,t}}, \varepsilon_{\xi_t,\delta_{4,t}}\big)\, \sigma\, dW_t,$$

where $\mu_{\xi,t} \equiv \hat{\mu}_{\xi,t}/\xi_t$ and $\varepsilon_{\xi_t,\delta_{j,t}} \equiv (1/\xi_t)\partial\xi_t/\partial\ln(\delta_{j,t}) = (\partial\xi_t/\partial\delta_{j,t})(\delta_{j,t}/\xi_t)$ is the elasticity of ξ_t with respect to $\delta_{j,t}$.

It follows from (A.17) that: $\varepsilon_{\xi_t,\delta_{1,t}} = (q-1)$, $\varepsilon_{\xi_t,\delta_{2,t}} = 0$,

$\varepsilon_{\xi_t,\delta_{3,t}} = (1-q)\{(\lambda_1)^{1/(q-1)}(\delta_{3,t})^{p/(1-q)}/[(\lambda_1)^{1/(q-1)}(\delta_{3,t})^{p/(1-q)}+(\lambda_2)^{1/(q-1)}(\delta_{4,t})^{p/(1-q)}]\}\, p/(1-q) = p\alpha_t$,

$\varepsilon_{\xi_t,\delta_{4,t}} = p(1-\alpha_t)$,

with $\alpha_t \equiv \{1+([\lambda_1(\delta_{4,t})^p]/[\lambda_2(\delta_{3,t})^p])^{1/(1-q)}\}^{-1} =$

$$(\lambda_1)^{1/(q-1)}(\delta_{3,t})^{p/(1-q)}/[(\lambda_1)^{1/(q-1)}(\delta_{3,t})^{p/(1-q)}+(\lambda_2)^{1/(q-1)}(\delta_{4,t})^{p/(1-q)}].$$

Note: (i) In the Comment, I define α_t as: $\alpha_t \equiv \{1+(\Lambda(\delta_{4,t}/\delta_{3,t})^p])^{1/(1-q)}\}^{-1}$, i.e. the term Λ in that formula corresponds to λ_1/λ_2.

(ii) Serrat's equation (14) (that defines α_t) contains a typo: in that equation $\delta_{1,t}$ has to be replaced by $\delta_{3,t}$.

Thus:

$$d\xi_t/\xi_t = \mu_{\xi,t}\, dt + \big(q-1,\ 0,\ p\alpha_t,\ p(1-\alpha_t)\big)\, \sigma\, dW_t.$$

This can be written as: $d\xi_t/\xi_t = \mu_{\xi,t}\, dt - \theta_t\, dW_t$, with $\theta_t \equiv \big(1-q,\ 0,\ -p\alpha_t,\ -p(1-\alpha_t)\big)\, \sigma$.

B. Discrete time model variant

In the discrete time version of Serrat's economy, the budget constraint is a difference equation in stock holdings (instead of Serrat's equation (4)), and lifetime utility is a weighted *sum* of instantaneous (period) utilities over the life cycle (instead of an integral of instantaneous utilities). The set of commodities, the instantaneous utility functions, and the asset structure are the same as in Serrat's model (see Sect. 2 of the Comment for a succinct description.)

B.1. Household decision problem

The economy starts in period $t=0$ and lasts until $T>0$. Country i $(i=1,2)$ is inhabited by a representative household whose budget constraint is:

$$\sum_{j=1}^{4} P_{j,t}S_{j,t+1}^{i} + \sum_{j=1}^{4} p_{j,t}c_{j,t}^{i} = \sum_{j=1}^{4} S_{j,t}^{i}(P_{j,t}+p_{j,t}\delta_{j,t}), \quad \text{for } 0 \le t \le T \qquad \text{(B.1)}$$

where $P_{j,t}$ is the (ex-dividend) price of stock *j* in period *t* (expressed in units of good 1); $S_{j,t+1}^{i}$ is the number of shares of stock *j* owned by country *i*, at the end of period *t* (beginning of t+1). See, e.g., Sargent (1987, pp.94-99) for a budget constraint of this type. Country *i*'s initial stock holdings (at the beginning of date 0) are exogenously given by $S_{1,0}^{i}, S_{2,0}^{i}, S_{3,0}^{i}, S_{4,0}^{i}$.

The total supply of each type of share is normalized at unity, i.e. $S_{j,t}^{i}=1$ represents 100% ownership of the "tree" that generates the endowment of good *j*. Recall that $p_{j,t}$ is the date *t* price of good *j*, and $\delta_{j,t}>0$ is the date *t* endowment of good *j*. Good 1 is used as a numéraire: $p_{1,t} \equiv 1$.

Country *i*'s preferences are described by $E_0\sum_{t=0}^{T}\beta^t V_t^i$, where V_t^i is *i*'s "instantaneous utility" at *t*, and where $0<\beta<1$ is the subjective discount factor. V_t^1 and V_t^2 are given by:

$$V_t^1=\tfrac{1}{q}(c_{3,t}^1)^p[(c_{1,t}^1)^q+(c_{2,t}^1)^q],\ V_t^2=\tfrac{1}{q}(c_{4,t}^2)^p[(c_{1,t}^2)^q+(c_{2,t}^2)^q], \text{ with } p+q<1,\ pq>0.$$

The decision variables of countries 1 and 2 at *t* are: $D_t^1=(S_{1,t+1}^1,S_{2,t+1}^1,S_{3,t+1}^1,S_{4,t+1}^1,c_{1,t}^1,c_{2,t}^1,c_{3,t}^1)$ and $D_t^2=(S_{1,t+1}^2,S_{2,t+1}^2,S_{3,t+1}^2,S_{4,t+1}^2,c_{1,t}^2,c_{2,t}^2,c_{4,t}^2)$, respectively.

The decision problem of country i is to select a process $\{D_t^i\}_{t=0}^T$ that maximizes $E_0\sum_{t=0}^{\infty}\beta^t V_t^i$ subject to (B.1) and to the following no-Ponzi-game condition:

$$\sum_{j=1}^{4} P_{j,T}S_{j,T+1}^{i} \ge 0.\ ^{1} \qquad \text{(B.2)}$$

The following equations are first-order conditions of the countries' decision problems:

$$1=E_t\rho_{t,t+1}^i(p_{j,t+1}\delta_{j,t+1}+P_{j,t+1})/P_{j,t} \quad \text{for } i=1,2;\ \ j=1,..,4;\ \ 0\le t\le T-1. \qquad \text{(B.3)}$$

$$\sum_{j=1}^{4} S_{j,T+1}^{i}P_{j,T} = 0 \ \text{ for } i=1,2. \qquad \text{(B.4)}$$

$$p_{2,t} = (c_{2,t}^i / c_{1,t}^i)^{q-1} \quad \text{for } i=1,2;\ \ 0\le t\le T. \qquad \text{(B.5)}$$

$$p_{3,t} = \tfrac{p}{q}(c_{1,t}^1/c_{3,t}^1)(1+(c_{2,t}^1/c_{1,t}^1)^q),\quad p_{4,t} = \tfrac{p}{q}(c_{1,t}^2/c_{4,t}^2)(1+(c_{2,t}^2/c_{1,t}^2)^q) \ \text{ for } 0\le t\le T. \qquad \text{(B.6)}$$

Here, $\rho_{t,t+s}^i$ (with $s\ge0$) is country i's marginal rate of substitution between consumption of good 1 at t and at $t+s$, for $0\le t,t+s\le T$:

$$\rho_{t,t+s}^1=\beta^s(c_{3,t+s}^1/c_{3,t}^1)^p(c_{1,t+s}^1/c_{1,t}^1)^{q-1} \ \text{ and } \ \rho_{t,t+s}^2=\beta^s(c_{4,t+s}^2/c_{4,t}^2)^p(c_{1,t+s}^2/c_{1,t}^2)^{q-1}.$$

---

[1] (B.2) ensure that the value of *i*'s life-time consumption process cannot exceed *i*'s initial wealth (see (B.17) below); this corresponds to Serrat's restriction that the consumption process has to be "admissible" (p.1472).

(B.3) shows i's Euler equations with respect to the 4 types of stocks. (B.4) follows from the complementary slackness (Kuhn-Tucker) condition associated with (B.2) (it cannot be optimal for country i to select $\sum_{j=1}^{4} P_{j,T} S_{j,T+1}^{i} > 0$, as this would imply that $i's$ consumption spending does not exhaust $i's$ resources). (B.5) and (B.6) say that country i equates her marginal rates of substitution between the goods that she consumes to the relative prices of these goods (the first [second] expression in (B.6) pertains to country 1 [2]).

B.2. Definition of competitive equilibrium

Serrat considers a competitive equilibrium (p.1473): in equilibrium, households maximize their expected life-time utility, subject to their budget constraints, taking prices as given; markets for goods and stocks clear. Given initial stock holdings $S_{1,0}^1, S_{2,0}^1, S_{3,0}^1, S_{4,0}^1, S_{1,0}^2 = 1 - S_{1,0}^1$, $S_{2,0}^2 = 1 - S_{2,0}^1, S_{3,0}^2 = 1 - S_{3,0}^1, S_{4,0}^2 = 1 - S_{4,0}^1$ an equilibrium is thus a process $\{c_{1,t}^1, c_{2,t}^1, c_{3,t}^1, c_{1,t}^2, c_{2,t}^2, c_{4,t}^2, p_{2,t}, p_{3,t}, p_{4,t}, P_{1,t}, P_{2,t}, P_{3,t}, P_{4,t}, S_{1,t+1}^1, S_{2,t+1}^1, S_{3,t+1}^1, S_{4,t+1}^1, S_{1,t+1}^2, S_{2,t+1}^2, S_{3,t+1}^2, S_{4,t+1}^2\}_{t=0}^{T}$ with these properties:

(i) (B.1) and (B.3)-(B.6) hold for $i = 1,2$.

(ii) Markets clear:

$$c_{1,t}^1 + c_{1,t}^2 = \delta_{1,t};\ c_{2,t}^1 + c_{2,t}^2 = \delta_{2,t};\ c_{3,t}^1 = \delta_{3,t};\ c_{4,t}^2 = \delta_{4,t};\ \ S_{j,t+1}^1 + S_{j,t+1}^2 = 1 \ \text{ for } j=1,..,4 \text{ and } 0 \le t \le T. \quad \text{(B.7)}$$

B.3. Efficient allocations

Serrat focuses on competitive equilibria that are Pareto efficient (i.e. that ensure full international risk sharing). An efficient allocation is the solution of this social planning problem:

$$Max \ \ (1-\lambda) E_0 \sum_{s=0}^{T} \beta^s V_s^1 + \lambda E_0 \sum_{s=0}^{T} \beta^s V_s^2 \ \text{ w.r.t. } \{c_{1,t}^1, c_{2,t}^1, c_{3,t}^1, c_{1,t}^2, c_{2,t}^2, c_{4,t}^2\}_{t=0}^{T}$$

$$\text{s.t. } c_{1,t}^1 + c_{1,t}^2 = \delta_{1,t},\ c_{2,t}^1 + c_{2,t}^2 = \delta_{2,t},\ c_{3,t}^1 = \delta_{3,t},\ c_{4,t}^2 = \delta_{4,t} \ \text{ at } 0 \le t \le T, \quad \text{(B.8)}$$

for some constant $0 \le \lambda \le 1$. [2] A key first-order condition of this problem is that marginal utilities of traded good consumption are perfectly correlated across countries:

$$(1-\lambda) \partial V_t^1 / \partial c_{j,t}^1 = \lambda \partial V_t^2 / \partial c_{j,t}^2 \ \ j=1,2 \ \text{ for } 0 \le t \le T. \quad \text{(B.9)}$$

In what follows, efficient consumptions are denoted by an asterisk. As *efficient* tradable good consumptions depend on $\Lambda \equiv \lambda/(1-\lambda)$ (see below), I write those consumptions as functions of Λ.

(B.8) and (B.9) imply that country 1 consumes a fraction

$$\alpha_t^*(\Lambda) \equiv \{1 + [\Lambda (\delta_{4,t}/\delta_{3,t})^p]^{\frac{1}{1-q}}\}^{-1},$$

of the world supply of tradables (see eqn. (13), (14) in Serrat):

$$c_{j,t}^{1*}(\Lambda) = \alpha_t^*(\Lambda) \delta_{j,t},\ c_{j,t}^{2*}(\Lambda) = (1-\alpha_t^*(\Lambda)) \delta_{j,t} \ \ (j=1,2), \text{ and } c_{3,t}^{1,*} = \delta_{1,t}, c_{4,t}^{2,*} = \delta_{2,t}, \text{ for } 0 \le t \le T. \quad \text{(B.10)}$$

[2] When $\lambda = 0$ or $\lambda = 1$, the social planning problem is trivial: one country consumes the entire world supply of tradables; the subsequent discussion assumes $0 < \lambda < 1$.

B.4. Decentralizing an efficient allocation

Proposition

Let $\{c_{1,t}^{1*}(\Lambda), c_{2,t}^{1*}(\Lambda), c_{3,t}^{1*}, c_{1,t}^{2*}(\Lambda), c_{2,t}^{2*}(\Lambda), c_{4,t}^{2*}\}_{t=0}^{T}$ be an efficient allocation, for some constant $\Lambda>0$. There exist goods prices, stock prices and stock holdings $\{p_{2,t}^{*}, p_{3,t}^{*}(\Lambda), p_{4,t}^{*}(\Lambda), P_{1,t}^{*}(\Lambda), P_{2,t}^{*}(\Lambda), P_{3,t}^{*}(\Lambda), P_{4,t}^{*}(\Lambda), S_{1,t+1}^{1*}, S_{2,t+1}^{1*}, S_{3,t+1}^{1*}, S_{4,t+1}^{1*}, S_{1,t+1}^{2*}, S_{2,t+1}^{2*}, S_{3,t+1}^{2*}, S_{4,t+1}^{2*}\}_{t=0}^{T}$, such that $\{c_{1,t}^{1*}(\Lambda), c_{2,t}^{1*}(\Lambda), c_{3,t}^{1*}, c_{1,t}^{2*}(\Lambda), c_{2,t}^{2*}(\Lambda), c_{4,t}^{2*}, p_{2,t}^{*}, p_{3,t}^{*}(\Lambda), p_{4,t}^{*}(\Lambda), P_{1,t}^{*}(\Lambda), P_{2,t}^{*}(\Lambda), P_{3,t}^{*}(\Lambda), P_{4,t}^{*}(\Lambda), S_{1,t+1}^{1*}, S_{2,t+1}^{1*}, S_{3,t+1}^{1*}, S_{4,t+1}^{1*}, S_{1,t+1}^{2*}, S_{2,t+1}^{2*}, S_{3,t+1}^{2*}, S_{4,t+1}^{2*}\}_{t=0}^{T}$ is a competitive equilibrium, for appropriate assignments of initial stock holdings $S_{1,0}^{1*}, S_{2,0}^{1*}, S_{3,0}^{1*}, S_{4,0}^{1*}, S_{1,0}^{2*}=1-S_{1,0}^{1*}, S_{2,0}^{2*}=1-S_{2,0}^{1*}, S_{3,0}^{2*}=1-S_{3,0}^{1*}, S_{4,0}^{2*}=1-S_{4,0}^{1*}$.

I denote all prices and stock holdings that pertain to an efficient competitive equilibrium by an asterisk; equilibrium prices that depend on Λ are written as functions of Λ.

In what follows, I show how to construct the process
$\{p_{2,t}^{*}, p_{3,t}^{*}(\Lambda), p_{4,t}^{*}(\Lambda), P_{1,t}^{*}(\Lambda), P_{2,t}^{*}(\Lambda), P_{3,t}^{*}(\Lambda), P_{4,t}^{*}(\Lambda), S_{1,t+1}^{1*}, S_{2,t+1}^{1*}, S_{3,t+1}^{1*}, S_{4,t+1}^{1*}, S_{1,t+1}^{2*}, S_{2,t+1}^{2*}, S_{3,t+1}^{2*}, S_{4,t+1}^{2*}\}_{t=0}^{T}$
and how to find appropriate initial stock holdings.

Goods prices

The goods prices $\{p_{2,t}^{*}, p_{3,t}^{*}(\Lambda), p_{4,t}^{*}(\Lambda)\}$ are found by substituting the efficient consumptions given in (B.10) into the countries' first-order conditions (B.5) and (B.6). This yields

$$p_{2,t}^{*}=(\delta_{2,t}/\delta_{1,t})^{q-1},\ p_{3,t}^{*}(\Lambda)=\tfrac{p}{q}(\alpha_t^{*}(\Lambda)/\delta_{3,t})Z_t,\ p_{4,t}^{*}(\Lambda)=\tfrac{p}{q}((1-\alpha_t^{*}(\Lambda))/\delta_{4,t})Z_t,\ Z_t\equiv\delta_{1,t}+(\delta_{1,t})^{1-q}(\delta_{2,t})^{q}, \quad \text{(B.11)}$$

where Z_t equals the value of the date t world endowment of tradables $\delta_{1,t}+p_{2,t}^{*}\delta_{2,t}$.

(B.9) implies that, in an efficient allocation, intertemporal marginal rates of substitution of tradable good consumption are equated across countries; let $\rho_{t,t+s}^{*}(\Lambda)$ denote the common marginal rate of substitution between consumption of good 1 at *t* and at $t+s$:

$$\rho_{t,t+s}^{*}(\Lambda) \equiv \rho_{t,t+s}^{1*}(\Lambda) = \rho_{t,t+s}^{2*}(\Lambda) \text{ for } s\geq 0,\ 0\leq t, t+s\leq T. \quad \text{(B.12)}$$

(B.10) implies:

$$\rho_{t,t+s}^{*}(\Lambda) = \beta^{s}(\delta_{3,t+s}/\delta_{3,t})^{p}([\alpha_{t+s}^{*}(\Lambda)\delta_{1,t+s}]/[\alpha_t^{*}(\Lambda)\delta_{1,t}])^{q-1}. \quad \text{(B.13)}$$

Stock prices

The date T stock prices have to satisfy (B.4): $\sum_{j=1}^{4} P_{j,T}^{*}S_{j,t+1}^{i*}=0$. Summing this condition across $i=1,2$ and using the market clearing condition for stocks $S_{j,T+1}^{1*}+S_{j,T+1}^{2*}=1$ (for $j=1,..,4$) gives: $\sum_{j=1}^{4} P_{j,T}^{*}=0$. If agents can freely dispose of "trees", equilibrium stocks prices cannot be negative. Thus, the (ex dividend) prices of all stocks are zero in the terminal period T:

$$P_{j,T}^{*}=0 \text{ for } j=1,..,4. \quad \text{(B.14a)}$$ [3]

Iterating (B.3) forward (using (B.14a), the common intertemporal marginal rate of substitution (B.13) and the goods prices defined in (B.11)) shows that stock prices satisfy this condition:

$$P_{j,t}^{*}(\Lambda)=E_t\sum_{s=1}^{T-t}\rho_{t,t+s}^{*}(\Lambda)p_{j,t+s}^{*}(\Lambda)\delta_{j,t+s} \quad \text{for } j=1,..,4,\ 0\leq t\leq T-1. \quad \text{(B.14b)}$$

[3] Hence, stock price bubbles are ruled out. In Serrat's equilibrium too, stock prices are zero at T.

The stock prices defined in (B.14a) and (B.14b), and both countries' efficient consumptions (B.10) satisfy the Euler equations (B.3); the goods prices (B.11) and the efficient consumptions (B.10) satisfy the first-order conditions (B.5) and (B.6); the efficient consumptions satisfy the market clearing conditions for goods listed in (B.7).

Stock holdings

To complete the proof of the **Proposition**, I have to find stock holdings $\{S^{1*}_{1,t+1}, S^{1*}_{2,t+1}, S^{1*}_{3,t+1}, S^{1*}_{4,t+1}, S^{2*}_{1,t+1}, S^{2*}_{2,t+1}, S^{2*}_{3,t+1}, S^{2*}_{4,t+1}\}_{t=-1}^{T}$ that are consistent with the budget constraint (B.1), with stock market clearing (see (B.7)), and with (B.4). Note that (B.4) holds for any values of $S^{i*}_{j,T+1}$, because $P^*_{j,T}=0$ for $j=1,..,4$.

Let $e^{i*}_t(\Lambda) \equiv \sum_{j=1}^{4} p^*_{j,t}(\Lambda) c^{i*}_{j,t}(\Lambda)$ denote the value of country i's efficient consumption basket (B.10), in period t, evaluated at the goods prices (B.11). (B.10) and (B.11) imply:

$$e^{1*}_t(\Lambda)=\alpha^*_t(\Lambda)(1+\tfrac{p}{q})Z_t\,,\quad e^{2*}_t(\Lambda)=(1-\alpha^*_t(\Lambda))(1+\tfrac{p}{q})Z_t. \qquad \text{(B.15)}$$

The stock holdings $\{S^{1*}_{1,t+1}, S^{1*}_{2,t+1}, S^{1*}_{3,t+1}, S^{1*}_{4,t+1}, S^{2*}_{1,t+1}, S^{2*}_{2,t+1}, S^{2*}_{3,t+1}, S^{2*}_{4,t+1}\}_{t=-1}^{T}$ thus have to satisfy the budget constraints

$$\sum_{j=1}^{4} S^{i*}_{j,t+1} P^*_{j,t}(\Lambda)+e^{i*}_t(\Lambda) = \sum_{j=1}^{4} S^{i*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)) \quad \text{for i=1,2 and } 0\leq t\leq T, \qquad \text{(B.16)}$$

as well as stock market clearing: $S^{1*}_{j,t+1}+S^{*}_{j,t+1}=1$ for $j=1,..,4$ and $-1\leq t\leq T$.

(B.16) holds if and only if the (ex dividend) value of the stocks held by country i at the beginning of t plus the date t dividend income generated by these stocks equals the value of i's efficient consumption expenditures $\{e^{i*}_{t+s}\}_{s=0}^{T-t}$ (evaluating using the pricing kernel $\rho^*_{t,t+s}$) :

$$E_t \sum_{s=0}^{T-t} \rho^*_{t,t+s}(\Lambda) e^{i*}_{t+s}(\Lambda) = \sum_{j=1}^{4} S^{i*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)) \quad \text{for } 0\leq t\leq T. \qquad \text{(B.17)}$$

● **Proof that (B.16) ⇒ (B.17)**

For $0\leq t\leq T$, let $W^{i*}_{t+1}(\Lambda)\equiv\sum_{j=1}^{4} P^*_{j,t}(\Lambda) S^{i*}_{j,t+1}$, and for $1\leq t\leq T-1$, $1+r^{i*}_t(\Lambda)\equiv\sum_{j=1}^{4} \frac{S^{i*}_{j,t}P^*_{j,t-1}(\Lambda)}{W^{i*}_t(\Lambda)} \frac{p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)}{P^*_{j,t-1}(\Lambda)}$.

$W^{i*}_{t+1}(\Lambda)$ is the value of i's equity portfolio at the end of t, and $1+r^{i*}_t(\Lambda)$ is the gross return on i's portfolio, between $t-1$ and t. (B.16) implies that $W^{i*}_{t+1}(\Lambda)+e^{i*}_t(\Lambda)=W^{i*}_t(\Lambda)(1+r^{i*}_t(\Lambda))$ for $1\leq t\leq T$. Therefore,

$$E_{t-1}\rho^*_{t-1,t}(\Lambda)(W^{i*}_{t+1}(\Lambda)+e^{i*}_t(\Lambda))=W^{i*}_t(\Lambda)E_{t-1}\rho^*_{t-1,t}(\Lambda)(1+r^{i*}_t(\Lambda)) \quad \text{for } 1\leq t\leq T.$$

As pointed out above, the stock prices $\{P^*_{j,t}(\Lambda)\}_{t=0}^{T}$ satisfy the Euler equations (B.3), and thus $1=E_{t-1}\rho^*_{t-1,t}(\Lambda)(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda))/P^*_{j,t-1}(\Lambda)$ for $1\leq t\leq T$. Hence, $1=E_{t-1}\rho^*_{t-1,t}(\Lambda)(1+r^{i*}_t(\Lambda))$, for $1\leq t\leq T$ (as can easily be shown using the above definition of $1+r^{i*}_t(\Lambda)$, and the fact that $S^{i*}_{j,t}$ and $W^{i*}_t(\Lambda)$ are in the date t-1 information set). Therefore,

$$E_{t-1}\rho^*_{t-1,t}(\Lambda)(W^{i*}_{t+1}(\Lambda)+e^{i*}_t(\Lambda))=W^{i*}_t(\Lambda) \text{ for } 1\le t\le T.$$

Iterating this equation forward, using the fact that $W^{i*}_{T+1}(\Lambda)=0$ (as $P^*_{j,T}(\Lambda)=0$ for $j=1,..,4$), gives

$$W^{i*}_t(\Lambda)=E_{t-1}\sum_{s=0}^{T-t}\rho^*_{t-1,t+s}(\Lambda)e^{i*}_{t+s}(\Lambda), \text{ for } 1\le t\le T.$$

Thus: $W^{i*}_{t+1}(\Lambda)=E_t\sum_{s=0}^{T-t-1}\rho^*_{t,t+1+s}(\Lambda)e^{i*}_{t+1+s}(\Lambda)$ for $0\le t\le T-1$. As $W^{i*}_{T+1}(\Lambda)=0$, this condition implies:

$$W^{i,*}_{t+1}(\Lambda)+e^{i*}_t(\Lambda)=E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)e^{i*}_{t+s}(\Lambda) \text{ for } 0\le t\le T. \quad \text{(B.18)}$$

(B.16) can be written as $W^{i*}_{t+1}(\Lambda)+e^{i*}_t(\Lambda)=\sum_{j=1}^{4}S^{i*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda))$ for $0\le t\le T$. The left-hand side of this equation can be replaced by $E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)e^{i*}_{t+s}(\Lambda)$ (because of (B.18)). This yields (B.17).

- **Proof that (B.17) $\Rightarrow$ (B.16)**

(B.17) implies

$$E_t\{\rho^*_{t,t+1}(\Lambda)\sum_{j=1}^{4}S^{i*}_{j,t+1}(p^*_{j,t+1}(\Lambda)\delta_{j,t+1}+P^*_{j,t+1}(\Lambda))\}+e^{i*}_t(\Lambda)=\sum_{j=1}^{4}S^{i*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)), \text{ for } 0\le t\le T-1.$$

Note that, $E_t\{\rho^*_{t,t+1}(\Lambda)\sum_{j=1}^{4}S^{i*}_{j,t+1}(p^*_{j,t+1}(\Lambda)\delta_{j,t+1}+P^*_{j,t+1}(\Lambda))\}=\sum_{j=1}^{4}S^{i*}_{j,t+1}P^*_{j,t}(\Lambda)$, as $S^{i*}_{j,t+1}$ is known at t and $P^*_{j,t}(\Lambda)=E_t\rho^*_{t,t+1}(\Lambda)(p^*_{j,t+1}(\Lambda)\delta_{j,t+1}+P^*_{j,t+1}(\Lambda))$ (Euler equation). Hence, (B.17) implies:

$$\sum_{j=1}^{4}S^{i*}_{j,t+1}P^*_{j,t}(\Lambda)+e^{i*}_t(\Lambda)=\sum_{j=1}^{4}S^{i*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)) \text{ for } 0\le t\le T-1. \quad \text{(B.19)}$$

In addition, (B.17) implies that $e^{i*}_T(\Lambda)=\sum_{j=1}^{4}S^{i*}_{j,T}p^*_{T,t}(\Lambda)\delta_{j,t}$; as $P^*_{j,T}=0$ for $j=1,..,4$, this gives $\sum_{j=1}^{4}S^{i*}_{j,T+1}P^*_{j,T}(\Lambda)+e^{i*}_T(\Lambda)=\sum_{j=1}^{4}S^{i*}_{j,T}(p^*_{j,T}(\Lambda)\delta_{j,T}+P^*_{j,T}(\Lambda))$. This and (B.19) implies (B.16).

When $\{S^{1*}_{1,t+1},S^{1*}_{2,t+1},S^{1*}_{3,t+1},S^{1*}_{4,t+1}\}_{t=-1}^{T}$ satisfies (B.17) for $i=1$, then $\{S^{2*}_{1,t+1},S^{2*}_{2,t+1},S^{2*}_{3,t+1},S^{2*}_{4,t+1}\}_{t=-1}^{T}$ with $S^{2*}_{j,t+1}=1-S^{1*}_{j,t+1}$ (for $j=1,..,4$) satisfies (B.17) for $i=2$, and vice versa. [4]

(Assume that (B.17) holds for $i=1$; substitute $S^{1*}_{j,t}=1-S^{2*}_{j,t}$ into (B.17) with $i=1$; this gives

$$E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)e^{1*}_{t+s}(\Lambda)=\sum_{j=1}^{4}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda))-\sum_{j=1}^{4}S^{2*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)). \quad \text{(B.20)}$$

Note that $E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)(e^{1*}_{t+s}(\Lambda)+e^{2*}_{t+s}(\Lambda))=E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)(\sum_{j=1}^{4}p^*_{j,t+s}(\Lambda)\delta_{j,t+s})=\sum_{j=1}^{4}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t})$, where the first equality follows from market clearing in goods markets, while the second equality follows from the stock price equations (B.14a) and (B.14b). Substituting this into (B.20) gives $E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)e^{2*}_{t+s}(\Lambda)=\sum_{j=1}^{4}S^{2*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda))$, which corresponds to (B.17) for $i=2$. Analogously, one can show that when (B.17) holds for $i=2$, then (B.17) holds for $i=1$.)

[4] I.e. when $\{S^{2*}_{1,t+1},S^{2*}_{2,t+1},S^{2*}_{3,t+1},S^{2*}_{4,t+1}\}_{t=-1}^{T}$ satisfies (B.17) for $i=2$, then $\{S^{1*}_{1,t+1},S^{1*}_{2,t+1},S^{1*}_{3,t+1},S^{1*}_{4,t+1}\}_{t=-1}^{T}$ with $S^{1*}_{j,t+1}=1-S^{2*}_{j,t+1}$ (for $j=1,..,4$) satisfies (B.17) for $i=1$.

Thus, the task of finding equilibrium stock holdings that finance the efficient allocation boils down to finding country 1 stock holdings $\{S^{1*}_{1,t+1}, S^{1*}_{2,t+1}, S^{1*}_{3,t+1}, S^{1*}_{4,t+1}\}^{T}_{t=-1}$ that satisfy (B.17) for i=1. The corresponding equilibrium stock holdings for country 2 are given by $S^{2*}_{j,t+1}$=1-$S^{1*}_{j,t+1}$ for j=1,..,4.

Finding stock holdings that satisfy (B.17), for i=1

The left-hand side of (B.17), for i=1, has the following property:

$$E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)e^{1*}_{t+s}(\Lambda)=(1+\tfrac{q}{p})(p^*_{3,t}(\Lambda)\delta_{3,t}+P^*_{3,t}(\Lambda)) \quad \text{for } 0\le t\le T. \tag{B.21}$$

(Note that (B.11) and (B.15) imply that $e^{1*}_t(\Lambda)=(1+\tfrac{q}{p})p^*_{3,t}\delta_{3,t}$. Thus

$$E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)e^{1*}_{t+s}(\Lambda)=(1+\tfrac{q}{p})E_t\sum_{s=0}^{T-t}\rho^*_{t,t+s}(\Lambda)p^*_{3,t+s}(\Lambda)\delta_{3,t+s}=(1+\tfrac{q}{p})(p^*_{3,t}(\Lambda)\delta_{3,t}+P^*_{3,t}(\Lambda)),$$

where the last equality follows from the stock price equations (B.14a) and (B.14b), for j=3.)

Using (B.21), one can write (B.17), for i=1, as follows:

$$(1+\tfrac{q}{p})(p^*_{3,t}(\Lambda)\delta_{3,t}+P^*_{3,t}(\Lambda))=\sum_{j=1}^{4}S^{1*}_{j,t}(p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)) \quad \text{for } 0\le t\le T. \tag{B.22}$$

As can easily be verified using (B.11), dividends (expressed in units of the numéraire) are collinear (see discussion in Comment):

$$p^*_{3,t}(\Lambda)\delta_{3,t}+p^*_{4,t}(\Lambda)\delta_{4,t}=\tfrac{p}{q}[\delta_{1,t}+p^*_{2,t}(\Lambda)\delta_{2,t}] \quad \text{for } 0\le t\le T. \tag{B.23}$$

It follows from (B.14a), (B.14b) and (B.23) that stock prices are likewise collinear:

$$P^*_{3,t}(\Lambda)+P^*_{4,t}(\Lambda)=\tfrac{p}{q}[P^*_{1,t}(\Lambda)+P^*_{2,t}(\Lambda)] \quad \text{for } 0\le t\le T. \tag{B.24}$$

(B.23) and (B.24) imply $\widetilde{P^*_{3,t}}(\Lambda)+\widetilde{P^*_{4,t}}(\Lambda)=\tfrac{p}{q}[\widetilde{P^*_{1,t}}(\Lambda)+\widetilde{P^*_{2,t}}(\Lambda)]$, with $\widetilde{P^*_{j,t}}(\Lambda)\equiv p^*_{j,t}(\Lambda)\delta_{j,t}+P^*_{j,t}(\Lambda)$.

(B.22) can be written as: $(1+\tfrac{q}{p})\widetilde{P^*_{3,t}}(\Lambda)=\sum_{j=1}^{4}S^{1*}_{j,t}\widetilde{P^*_{j,t}}(\Lambda)$ for $0\le t\le T$. Substituting $\widetilde{P^*_{4,t}}(\Lambda)=\tfrac{p}{q}[\widetilde{P^*_{1,t}}(\Lambda)+\widetilde{P^*_{2,t}}(\Lambda)]-\widetilde{P^*_{3,t}}(\Lambda)$ into this expression yields:

$$0=(S^{1*}_{1,t}+\tfrac{p}{q}S^{1*}_{4,t})\widetilde{P^*_{1,t}}(\Lambda)+(S^{1*}_{2,t}+\tfrac{p}{q}S^{1*}_{4,t})\widetilde{P^*_{2,t}}(\Lambda)+(S^{1*}_{3,t}-S^{1*}_{4,t}-1-\tfrac{q}{p})\widetilde{P^*_{3,t}}(\Lambda), \tag{B.25}$$

for $0\le t\le T$.

Thus, **any** process $\{S^{1,*}_{1,t+1},S^{1,*}_{2,t+1},S^{1,*}_{3,t+1},S^{1,*}_{4,t+1},S^{2,*}_{1,t+1},S^{2,*}_{2,t+1},S^{2,*}_{3,t+1},S^{2,*}_{4,t+1}\}^{T}_{t=-1}$ that satisfies (B.25) and $S^{2*}_{j,t+1}=1-S^{1*}_{j,t+1}$, for j=1,..,4 and $-1\le t\le T$, is consistent with (B.16) (and thus with the budget constraints (B.1) for i=1,2), with (B.4), and with the market clearing condition for stocks listed in (B.7). (As mentioned above, (B.4) holds for arbitrary values of $S^{i*}_{j,T+1}$ because $P^*_{j,T}(\Lambda)=0$ for j=1,..,4.)

(B.25) holds if $0=S^{1*}_{1,t}+\tfrac{p}{q}S^{1*}_{4,t}=S^{1*}_{2,t}+\tfrac{p}{q}S^{1*}_{4,t}=S^{1*}_{3,t}-S^{1*}_{4,t}-1-\tfrac{q}{p}$ or, equivalently, if

$$S^{1*}_{2,t}=S^{1*}_{1,t},\ S^{1*}_{3,t}=1+\tfrac{q}{p}-\tfrac{q}{p}S^{1*}_{1,t},\ S^{1*}_{4,t}=-\tfrac{q}{p}S^{1*}_{1,t}. \tag{B.26}$$

The Comment (see eqn. (3)) shows that the time-invariant share holdings $S^1_2=S^1_1$, $S^1_3=1+\tfrac{q}{p}-\tfrac{q}{p}S^1_1$, $S^1_4=-\tfrac{q}{p}S^1_1$ support the efficient equilibrium; these share holdings are consistent with (B.26).

Conclusion:
For any $\Lambda>0$, the Pareto efficient consumptions, the goods prices and stock prices, and the stock holdings $\{c_{1,t}^{1*}(\Lambda),c_{2,t}^{1*}(\Lambda),c_{3,t}^{1*},c_{1,t}^{2*}(\Lambda),c_{2,t}^{2*}(\Lambda),c_{4,t}^{2*},p_{2,t}^{*},p_{3,t}^{*}(\Lambda),p_{4,t}^{*}(\Lambda),P_{1,t}^{*}(\Lambda),P_{2,t}^{*}(\Lambda),P_{3,t}^{*}(\Lambda),P_{4,t}^{*}(\Lambda),$ $S_{1,t+1}^{1*},S_{2,t+1}^{1*},S_{3,t+1}^{1*},S_{4,t+1}^{1*},S_{1,t+1}^{2*},S_{2,t+1}^{2*},S_{3,t+1}^{2*},S_{4,t+1}^{2*}\}_{t=0}^{T}$ defined by (B.10), (B.11), (B.14a), (A14b) and (B.26) (for $0\le t\le T$) and $S_{j,t+1}^{1*}+S_{j,t+1}^{2*}=1$ (for $j=1,..,4$ and $0\le t\le T$) are a competitive equilibrium, relative to initial stock holdings $S_{1,0}^{1*},S_{2,0}^{1*},S_{3,0}^{1*},S_{4,0}^{1*},S_{1,0}^{2*},$ $S_{2,0}^{2*},S_{3,0}^{2*},S_{4,0}^{2*}$ that satisfy (B.25) (for $t=0$) and $S_{j,0}^{1*}+S_{j,0}^{2*}=1$ for $j=1,..,4$.

This completes the proof of **Proposition 1**.

Remarks:

1) The preceding discussion shows that the equilibrium goods prices and stock prices that support an efficient allocation are unique; by contrast, stock holdings are indeterminate.

2) Clearly, (B.26) is a sufficient conditions that ensures that (B.25) holds. In addition, it appears that, if the covariance matrix of endowment innovations is non-singular (as assumed by Serrat), (B.26) for $1\le t\le T$ is a **necessary** condition under which (B.25) holds (for all states of the world at) $1\le t\le T$. **Thus, when (B.26) does not hold for $1\le t\le T$, then (B.25) cannot hold (for all states of the world) at $1\le t\le T$.**

Note that (B.25) holds for $1\le t\le T$, then

$$0=(S_{1,t}^{1*}+\tfrac{p}{q}S_{4,t}^{1*})\varepsilon_{1,t}^{*}(\Lambda)+(S_{2,t}^{1*}+\tfrac{p}{q}S_{4,t}^{1*})\varepsilon_{2,t}^{*}(\Lambda)+(S_{3,t}^{1*}-S_{4,t}^{1*}-1-\tfrac{q}{p})\varepsilon_{3,t}^{*}(\Lambda) \quad \text{for } 1\le t\le T, \qquad \text{(B.27)}$$

where $\varepsilon_{j,t}^{*}(\Lambda)\equiv\widetilde{P_{j,t}^{*}}(\Lambda)-E_{t-1}\widetilde{P_{j,t}^{*}}(\Lambda)$. The *three* innovations $\varepsilon_{1,t}^{*}(\Lambda),\varepsilon_{2,t}^{*}(\Lambda),\varepsilon_{3,t}^{*}(\Lambda)$ are functions of innovations to the *four* endowments $\delta_{1,t},\delta_{2,t},\delta_{3,t},\delta_{4,t}$. If the covariance matrix of the four endowment innovations is non-singular, $\varepsilon_{1,t}^{*}(\Lambda),\varepsilon_{2,t}^{*}(\Lambda)$ and $\varepsilon_{3,t}^{*}(\Lambda)$ are not collinear (see discussion in Sect B.6).

[In the continuous time model, the counterpart to this lack of collinearity is the fact that the first three rows of the diffusion matrix of stock prices σ_t^G shown in Serrat's eqn. (16) are linearly independent, and that the first three columns of the matrix Λ_t shown on p.1483 are linearly independent.]

As $S_{j,t}^{1*}$ is set at $t-1$, this lack of collinearity implies that (B.27) holds for random realizations of $\varepsilon_{1,t}^{*}(\Lambda),\varepsilon_{2,t}^{*}(\Lambda),\varepsilon_{3,t}^{*}(\Lambda)$ if and only if

$$0=S_{1,t}^{1*}+\tfrac{p}{q}S_{4,t}^{1*}=S_{2,t}^{1*}+\tfrac{p}{q}S_{4,t}^{1*}=S_{3,t}^{1*}-S_{4,t}^{1*}-1-\tfrac{q}{p} \text{ for } 1\le t\le T,$$

in other terms iff (B.26) holds for $1\le t\le T$.

Hence, stock holdings that do not satisfy (B.26) for $1\le t\le T$ cannot satisfy (B.25) (and (B.1)) for all states of the world at $1\le t\le T$. Thus, stock holdings that do not satisfy (B.26) for $1\le t\le T$ do not implement an efficient competitive equilibrium.

3) The equilibrium portfolios in the discrete time variant have the same structure as portfolios in the (correctly solved) continuous time model, as can be seen by comparing (B.26) and equations (3) and (5) in the Comment.

4) The intuition for the preceding point is that the key equilibrium conditions for portfolios are very similar, in the continuous and discrete time variants.

Note that (B.22) implies that (in discrete time)

$$\sum_{j=1}^{4} S_{j,t}^{1*}\varepsilon_{j,t}^{*}(\Lambda) = (1+\tfrac{q}{p})\varepsilon_{3,t}^{*}(\Lambda) \text{ for } 1\leq t\leq T. \tag{B.28}$$

This condition is related to Serrat's key portfolio equation (continuous time), reproduced as eqn. (4) in the Comment:

$$\Lambda_t \pi_t^{i}{}' = \widehat{\Phi}_t^{i}{}', \quad \text{with } \pi_t^i \equiv (\pi_{1,t}^i, \pi_{2,t}^i, \pi_{3,t}^i, \pi_{4,t}^i),\ \pi_{j,t}^i \equiv S_{j,t}^i P_{j,t}. \tag{4}$$

(in Serrat's paper, see last line of p.1484, as well as equations (7) and (34)).

As shown in Sections A.2.2 and A.2.3, $\widehat{\Phi}_t^{1}{}' = \lambda_{3,t}(1+\tfrac{q}{p})P_{3,t}$, where $\lambda_{3,t}$ is the third column of Λ_t. For i=1, (4) can thus be written as $\Lambda_t \pi_t^{1}{}' = \lambda_{3,t}(1+\tfrac{q}{p})P_{3,t}$. As σ (diffusion matrix of the vector of (log) endowments) is assumed to be non-singular (p.1470), this expression is equivalent to: $\pi_t^1 \Lambda_t{}' \sigma = (1+\tfrac{q}{p})P_{3,t}\lambda_{3,t}{}'\sigma$. Note that $\sigma_t^G \equiv \Lambda_t{}'\sigma$, where σ_t^G is the diffusion matrix of stock returns (see p.1483 and Serrat's eqn. (2)). Thus: $\pi_t^1 \sigma_t^G = (1+\tfrac{q}{p})P_{3,t}\sigma_{3,t}^G$, where $\sigma_{3,t}^G$ is the third row of σ_t^G. This equation implies

$$\pi_t^1 \sigma_t^G dW_t = (1+\tfrac{q}{p})P_{3,t}\sigma_{3,t}^G dW_t, \tag{B.29}$$

where W_t is the four-dimensional Wiener process that drives the logged endowments (see Serrat's eqn. (1)). Serrat's eqn. (2) implies that $\sigma_t^G dW_t$ equals the vector of stock return innovations, and can be written as:

$$\sigma_t^G dW_t = ([dP_{1,t} - E_t dP_{1,t}]/P_{1,t}, [dP_{2,t} - E_t dP_{2,t}]/P_{2,t}, [dP_{3,t} - E_t dP_{3,t}]/P_{3,t}, [dP_{4,t} - E_t dP_{4,t}]/P_{4,t})'.$$

As $\pi_{j,t}^1 \equiv S_{j,t}^1 P_{j,t}$, (B.29) can hence be expressed as:

$$\sum_{j=1}^{4} S_{j,t}^1 [dP_{j,t} - E_t dP_{j,t}] = (1+\tfrac{q}{p})[dP_{3,t} - E_t dP_{3,t}],$$

which closely resembles eqn. (B.28) for the discrete time variant: in both the continuous and discrete time structures, equilibrium stockholdings entail that innovations to the value of country 1's total portfolio (left-hand side) track the innovations to the present value of country 1's efficient consumption spending; see right-had side (NB that present value equals $(1+\tfrac{q}{p})P_{3,t}$).

5) Serrat (Theorem 2) claims that in equilibrium claims to domestic non-tradables are only held by domestic investors: $S_{3,t}^1=1, S_{4,t}^1=0$ for $t>0$. These stock holdings are inconsistent with (B.26). Thus, Serrat's portfolio is incompatible with an efficient equilibrium (it does not finance the efficient consumptions defined by Serrat's equations (13)-(14).)

B.5. Characterizing efficient equilibria for exogenous initial stock holdings

Serrat assumes that initial stock holdings are given by:

$$S^1_{1,0}{=}S^1_{3,0}{=}1; S^1_{2,0}{=}S^1_{4,0}{=}0\,. \tag{B.30}$$

These initial holdings are inconsistent with (B.26). Nevertheless, an efficient competitive equilibrium exists, relative to these initial share holdings. There exists a unique value of Λ such that the expression in (B.25) holds for $t{=}0$, i.e. such that the value of each country's initial equity portfolio equals the value of its efficient consumption process.

Evaluating (B.25) for $t{=}0$ and share holdings (B.30) gives:

$$0{=}\widetilde{P^*_{1,0}}(\Lambda)-\widetilde{P^*_{3,0}}(\Lambda)\tfrac{q}{p}.$$

Note that $\widetilde{P^*_{j,0}}(\Lambda){=}E_0\sum_{s=0}^{T}\rho^*_{0,s}(\Lambda)p^*_{j,s}(\Lambda)\delta_{j,s}$. Thus (from (B.11) and (B.13)):

$\widetilde{P^*_{1,0}}(\Lambda) = E_0\sum_{s=0}^{T}\beta^s(\delta_{3,s}/\delta_{3,0})^p\{[\alpha^*_s(\Lambda)\delta_{1,s}]/[\alpha^*_0(\Lambda)\delta_{1,0}]\}^{q-1}\delta_{1,s}$ and

$$\widetilde{P^*_{3,0}}(\Lambda) = E_0\sum_{s=0}^{T}\beta^s(\delta_{3,s}/\delta_{3,0})^p\{[\alpha^*_s(\Lambda)\delta_{1,s}]/[\alpha^*_0(\Lambda)\delta_{1,0}]\}^{q-1}\tfrac{p}{q}\alpha^*_s(\Lambda)[\delta_{1,s}+(\delta_{1,s})^{1-q}(\delta_{2,s})^q].$$

$\alpha^*_t(\Lambda)$ is a continuous function of Λ. Hence $\widetilde{P^*_{1,0}}(\Lambda), \widetilde{P^*_{3,0}}(\Lambda)$ are likewise continuous functions of Λ. Note that $\alpha^*_s(0){=}1, \alpha^*_s(\infty){=}0$, $\alpha^*_s(\infty)/\alpha^*_0(\infty){=}([\delta_{3,s}/\delta_{4,s}]/[\delta_{3,0}/\delta_{4,0}])^{p/(1-q)}$. Thus, $\widetilde{P^*_{1,0}}(0)-\widetilde{P^*_{3,0}}(0)\tfrac{q}{p}{<}0$ and $\widetilde{P^*_{1,0}}(\infty)-\widetilde{P^*_{3,0}}(\infty)\tfrac{q}{p}{>}0$. Hence, there exists a value $\widehat{\Lambda}$ for which $0{=}\widetilde{P^*_{1,0}}(\widehat{\Lambda})-\widetilde{P^*_{3,0}}(\widehat{\Lambda})\tfrac{q}{p}$. It can be shown that $d\{\widetilde{P^*_{1,0}}(\Lambda)-\widetilde{P^*_{3,0}}(\Lambda)\tfrac{q}{p}\}/d\Lambda\ >0$ at $\Lambda{=}\widehat{\Lambda}$. Thus $\widehat{\Lambda}$ is unique.

For $\widehat{\Lambda}$ that satisfies $0{=}\widetilde{P^*_{1,0}}(\widehat{\Lambda})-\widetilde{P^*_{3,0}}(\widehat{\Lambda})\tfrac{q}{p}$, the process $\{c^{1*}_{1,t}(\widehat{\Lambda}), c^{1*}_{2,t}(\widehat{\Lambda}), c^{1*}_{3,t}, c^{2*}_{1,t}(\widehat{\Lambda}), c^{2*}_{2,t}(\widehat{\Lambda}), c^{2*}_{4,t}$, $p^*_{2,t}, p^*_{3,t}(\widehat{\Lambda}), p^*_{4,t}(\widehat{\Lambda}), P^*_{1,t}(\widehat{\Lambda}), P^*_{2,t}(\widehat{\Lambda}), P^*_{3,t}(\widehat{\Lambda}), P^*_{4,t}(\widehat{\Lambda}), S^{1*}_{1,t+1}, S^{1*}_{2,t+1}, S^{1*}_{3,t+1}, S^{1*}_{4,t+1}, S^{2*}_{1,t+1}, S^{2*}_{2,t+1}, S^{2*}_{3,t+1}, S^{2*}_{4,t+1}\}^T_{t=0}$ defined by (B.10), (B.11), (B.14a), (B.14b) and by (B.26) (for $1{\le}t\le T$) with $\Lambda{=}\widehat{\Lambda}$ is an efficient competitive equilibrium, with respect to the initial stock holdings (B.30).

Summary: the initial portfolio pins down the weight Λ that determines what share of the world supply of tradables is consumed by country 1 (α_t); in the efficient competitive equilibrium, countries rebalance their portfolios at $t{=}0$ so that

$$S^1_{2,t} = S^1_{1,t}\,,\ S^1_{3,t} = 1+\tfrac{q}{p}-\tfrac{q}{p}S^1_{1,t}\,,\ S^1_{4,t} = -\tfrac{q}{p}S^1_{1,t} \quad \text{for } 1{\le}t{\le}T.$$

Remark: the determination of Λ presented here corresponds to that used by Serrat. Serrat solves for Λ (which in his notation corresponds to λ_1/λ_2; see his eqn. (14)) by solving his eqn. (15)--he thus sets Λ at the value that ensures that the value of each country's initial equity portfolio equals the value of its efficient consumption process.

B.6. Innovations to $\widetilde{P^*_{1,t}}(\Lambda)$, $\widetilde{P^*_{2,t}}(\Lambda)$, $\widetilde{P^*_{3,t}}(\Lambda)$ are not collinear

$\widetilde{P^*_{1,t}}(\Lambda), \widetilde{P^*_{2,t}}(\Lambda)$ and $\widetilde{P^*_{3,t}}(\Lambda)$ are functions of the exogenous endowments of the *four* goods. (NB $\widetilde{P^*_{j,t}}(\Lambda) \equiv p^*_{j,t}(\Lambda)\delta_{j,t} + P^*_{j,t,}(\Lambda)$.) If the covariance matrix of the endowment innovations is non-singular (as assumed by Serrat, p.1470), then innovations to $\widetilde{P^*_{1,t}}(\Lambda), \widetilde{P^*_{2,t}}(\Lambda), \widetilde{P^*_{3,t}}(\Lambda)$ are not collinear. I now illustrate this (rather obvious) point, using assumptions about endowments that yield simpler solutions for stock prices than Serrat's assumptions.

Equations (B.11), (B.13), (B.14a) and (B.14b) in Section B.4 imply

$$\widetilde{P^*_{1,t}}(\Lambda) = E_t \sum_{s=0}^{T-t} \beta^s (\delta_{3,t+s}/\delta_{3,t})^p \{[\alpha^*_{t+s}(\Lambda)\delta_{1,t+s}]/[\alpha^*_t(\Lambda)\delta_{1,t}]\}^{q-1}\delta_{1,t+s},$$

$$\widetilde{P^*_{2,t}}(\Lambda) = E_t \sum_{s=0}^{T-t} \beta^s (\delta_{3,t+s}/\delta_{3,t})^p \{[\alpha^*_{t+s}(\Lambda)\delta_{1,t+s}]/[\alpha^*_t(\Lambda)\delta_{1,t}]\}^{q-1}(\delta_{1,t+s})^{1-q}(\delta_{2,t+s})^q,$$

$$\widetilde{P^*_{3,t}}(\Lambda) = E_0 \sum_{s=0}^{T-t} \beta^s (\delta_{3,t+s}/\delta_{3,t})^p \{[\alpha^*_{t+s}(\Lambda)\delta_{1,t+s}]/[\alpha^*_t(\Lambda)\delta_{1,t}]\}^{q-1}\tfrac{p}{q}\alpha^*_{t+s}(\Lambda)[\delta_{1,t+s} + (\delta_{1,t+s})^{1-q}(\delta_{2,t+s})^q],$$

for $0 \le t \le T$.

Thus, $\widetilde{P^*_{1,t}}(\Lambda), \widetilde{P^*_{w,t}}(\Lambda)$ and $\widetilde{P^*_{3,t}}(\Lambda)$ are functions of $\{\delta_{1,t+s}, \delta_{2,t+s}, \delta_{3,t+s}, \alpha^*_{t+s}(\Lambda)\}_{s\ge 0}$. Assume that $\ln\delta_{1,t}$, $\ln\delta_{2,t}$, $\ln\delta_{3,t}$, $\ln\alpha^*_t(\Lambda)$ follow random walks without drift:

$\ln\delta_{1,t} - \ln\delta_{1,t-1} = s_1\eta_{1,t}$, $\ln\delta_{2,t} - \ln\delta_{2,t-1} = s_2\eta_{2,t}$, $\ln\delta_{3,t} - \ln\delta_{3,t-1} = s_3\eta_{3,t}$, $\ln\alpha^*_t(\Lambda) - \ln\alpha^*_{t-1}(\Lambda) = s_\alpha\eta_{\alpha,t}$,

where s_1, s_2, s_3, s_α are constants and $\eta_{1,t}$, $\eta_{2,t}$, $\eta_{3,t}$, $\eta_{\alpha,t}$ are independent $N(0,1)$ white noises. [5] Then,

$$\widetilde{P^*_{1,t}}(\Lambda) = K_1\delta_{1,t}, \quad \widetilde{P^*_{2,t}}(\Lambda) = K_2(\delta_{1,t})^{1-q}(\delta_{2,t})^q, \quad \widetilde{P^*_{3,t}}(\Lambda) = K_3\alpha^*_t(\Lambda)\delta_{1,t} + K_4\alpha_t(\Lambda)(\delta_{1,t})^{1-q}(\delta_{2,t})^q,$$

where K_1, K_2, K_3, K_4 are constants.

Let $\varepsilon^*_{j,t}(\Lambda) \equiv \widetilde{P^*_{j,t}}(\Lambda) - E_{t-1}\widetilde{P^*_{j,t}}(\Lambda)$. We have: $\varepsilon^*_{1,t}(\Lambda) = \widetilde{P^*_{1,t-1}}(\Lambda)\{\exp(s_1\eta_{1,t}) - \exp(\tfrac{1}{2}(s_1)^2)\}$,

$$\varepsilon^*_{2,t}(\Lambda) = \widetilde{P^*_{2,t-1}}(\Lambda)\{\exp[(1-q)s_1\eta_{1,t} + qs_2\eta_{2,t}] - \exp[\tfrac{1}{2}(1-q)^2(s_1)^2 + \tfrac{1}{2}q^2(s_2)^2]\},$$

$$\varepsilon^*_{3,t}(\Lambda) = K_3\alpha_{3,t-1}\delta_{1,t-1}\{\exp[s_\alpha\eta_{\alpha,t} + s_1\eta_{1,t}] - \exp[\tfrac{1}{2}(s_\alpha)^2 + \tfrac{1}{2}(s_1)^2]\} +$$
$$K_4\alpha^*_{t-1}(\Lambda)(\delta_{1,t-1})^{1-q}(\delta_{2,t-1})^q[\exp\{s_\alpha\eta_{\alpha,t} + (1-q)s_1\eta_{1,t} + qs_2\eta_{2,t}\} - \exp\{\tfrac{1}{2}[(s_\alpha)^2 + (1-q)^2(s_1)^2 + q^2(s_2)^2]\}].$$

These formulae show clearly that $\varepsilon^*_{1,t}(\Lambda), \varepsilon^*_{2,t}(\Lambda), \varepsilon^*_{3,t}(\Lambda)$ are not collinear. (Note that $\varepsilon^*_{1,t}(\Lambda)$ is a function of $\eta_{1,t}$; $\varepsilon^*_{2,t}(\Lambda)$ is a function of $\eta_{1,t}$ and $\eta_{2,t}$; $\varepsilon^*_{3,t}(\Lambda)$ is a function of $\eta_{1,t}$, $\eta_{2,t}$ and $\eta_{\alpha,t}$.)

[5] In the economic model, $\alpha^*_t(\Lambda)$ cannot exceed unity. Violations of that upper bound can be ruled out by assuming that $\alpha^*_0(\Lambda) < 1$ and by setting the variance of innovations to $\alpha^*_t(\Lambda)$ $((s_\alpha)^2)$ at a sufficiently small value. Serrat assumes that the four logged endowments follow random walks; this yields solutions for stock prices that are much more complicated than the solutions shown below.

References

Miranda, M., Fackler, P. (2002): Applied Computational Economics and Finance, MIT Press: Cambridge, MA.

Sargent, T. (1987): *Dynamic Macroeconomic Theory*, Harvard University Press: Cambridge, MA.