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BLANCHARD AND KAHN'S (1980) SOLUTION FOR A LINEAR RATIONAL EXPECTATIONS MODEL WITH ONE STATE VARIABLE AND ONE JUMP VARIABLE: THE CORRECT FORMULA*

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This note corrects Blanchard and Kahn's (1980) formula for the solution of a linear dynamic rational expectations model with one predetermined and one non-predetermined endogenous variable.

1 Introduction

In their classical paper, Blanchard and Kahn (1980) [BK] derived the solution for an important class of dynamic linear rational expectations models. The BK algorithm has become a standard tool for economic modelers.[1] In general, the model solution is analytically intractable. However, as shown by BK, a model with one predetermined and one non-predetermined endogenous variable can be handled analytically, which facilitates an intuitive understanding of the solution. That special case is important (and of pedagogical interest) as it includes, e.g., the basic RBC model with fixed labor. This note shows that the formula provided by BK, for this special case, includes an error; we also provide the correct formula.[2]

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[1]The BK algorithm is, e.g., often used to solve linearized dynamic stochastic general equilibrium models, the workhorses of modern macroeconomics (King and Rebelo, 1999; Schmitt-Grohé and Uribe, 2004; Kollmann, 2015). Google Scholar records 2389 cites (09/2016) for the BK paper.

[2]Most likely this error is a typing or printing error. BK is a very classical paper, so a correct version of all key results in it should be available. BK's *general* routine for models with arbitrary numbers of states and jump variables (widely used in computer simulation packages) is accurate.

2 A Linear Dynamic Model with One State Variable and One Jump Variable

Consider the following model (the notation follows BK):

$$\begin{bmatrix} x_{t+1} \\ E_t p_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} Z_t, \tag{1}$$

where x_t is a predetermined variable ('state'), and p_t is a non-predetermined ('jump') variable. Z_t is a $(k \times 1)$ vector of exogenous variables. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a (2×2) matrix, and γ_1, γ_2 are $(1 \times k)$ vectors. Let λ_1, λ_2 be the eigenvalues of A, with $|\lambda_1| \leq |\lambda_2|$, and let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be the matrix of eigenvectors of A, i.e. $AB = BJ$, with $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Let $C \equiv B^{-1}$, $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$. Note that $A = BJC$. Proposition 1 of BK (p.1308) shows that model (1) has a unique (non-exploding) solution if and only if $|\lambda_1| \leq 1$, $|\lambda_2| > 1$. BK (p.1309) state that then the solution of (1) is:

$$x_t = \lambda_1 x_{t-1} + \gamma_1 Z_{t-1} + \mu \sum\nolimits_{i=0}^{\infty} \lambda_2^{-i-1} E_{t-1} Z_{t+i-1}, \tag{2}$$

$$p_t = a_{12}^{-1} \left[(\lambda_1 - a_{11}) x_t + \mu \sum\nolimits_{i=0}^{\infty} \lambda_2^{-i-1} E_t Z_{t+i} \right], \tag{3}$$

$$\text{with } \mu \equiv (\lambda_1 - a_{11}) \lambda_1 - a_{12} \lambda_2. \tag{4}$$

Comment: When μ is given by (4), then $\mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_{t-1} Z_{t+i-1}$ and $\mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_t Z_{t+i}$ are $(k \times 1)$ vectors; thus (2), (3) cannot hold for $k > 1$ as x_t, p_t are scalars. As shown below BK's formula (4) for μ is incorrect. The correct formula is:

$$\mu = (\lambda_1 - a_{11}) \gamma_1 - a_{12} \gamma_2. \tag{5}$$

Proof: (2) and (3) are special cases of the general solution for linear difference models (with arbitrary numbers of variables) in BK's Proposition 1. (See Appendix.) The general solution for state x_t shows that the correct expression for μ in (2) is $\mu = -(b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22}) c_{22}^{-1} (c_{21}\gamma_1 + c_{22}\gamma_2)$. Write this as $\mu = \phi_1 \gamma_1 + \phi_2 \gamma_2$, with $\phi_1 \equiv -(b_{11}\lambda_1 c_{12} c_{22}^{-1} c_{21} + b_{12}\lambda_2 c_{21})$, $\phi_2 = -(b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22})$. $A = BJC$ implies $a_{11} = b_{11}\lambda_1 c_{11} + b_{12}\lambda_2 c_{21}$ and $a_{12} = b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22}$. Thus $\phi_2 = -a_{12}$. Substituting $b_{12}\lambda_2 c_{21} = a_{11} - b_{11}\lambda_1 c_{11}$ into the definition of ϕ_1 gives $\phi_1 \equiv -(a_{11} + b_{11}\lambda_1 [c_{12} c_{22}^{-1} c_{21} - c_{11}])$. $B = C^{-1}$ implies $b_{11} = c_{22}/(c_{11}c_{22} - c_{12}c_{21})$ and $c_{12} c_{22}^{-1} c_{21} - c_{11} = -b_{11}^{-1}$. Thus

$\phi_1 = \lambda_1 - a_{11}$. This implies (5). The general solution for jump variable p_t shows that equation 3 holds when μ is defined by (5). ■

Appendix

Blanchard and Kahn (1980): the general model

Consider the model

$$\begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t, \qquad \text{(A1)}$$

where X_t is an $n \times 1$ vector of predetermined variable, and p_t is an $m \times 1$ vector of non-predetermined variables; Z_t is a $(k \times 1)$ vector of exogenous variables. A is an $(n+m) \times (n+m)$ matrix, and γ is an $(n+m) \times k$ matrix. Consider the Jordan canonical form $A = C^{-1}JC$, where C and J are $(n+m) \times (n+m)$ matrices. Let the diagonal elements of J (i.e. the eigenvalues of A) be ordered by increasing absolute value. Let $\overline{n}$ ($\overline{m}$) denote the number of eigenvalues of A that are on or inside the unit circle (outside the unit circle). Partition J as $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, where J_1 and J_2 are matrices of dimensions ($\overline{n}$x$\overline{n}$) and ($\overline{m}$x$\overline{m}$) respectively. Decompose C, $B \equiv C^{-1}$ and γ as $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ and $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$, where $C_{11}, C_{12}, C_{21}, C_{22}$ are matrices of dimensions ($\overline{n}$xn), ($\overline{n}$xm), ($\overline{m}$xn) and ($\overline{m}$xm) respectively; $B_{11}, B_{12}, B_{21}, B_{22}$ have dimensions (nx$\overline{n}$), (nx$\overline{m}$), (mx$\overline{n}$) and (mx$\overline{m}$) respectively, while γ_1 and γ_2 have dimensions ($\overline{n}$xk) and ($\overline{m}$xk) respectively. Proposition 1 in Blanchard and Kahn (1980) states that the model (A1) has a unique (non-explosive) solution if and only if the number of non-predetermined variables equals the number of eigenvalues of A outside the unit circle: $m = \overline{m}$. If that condition is met, then the solution is:

$$X_t = B_{11} J_1 B_{11}^{-1} X_{t-1} + \gamma_1 Z_{t-1} - (B_{11} J_1 C_{12} + B_{12} J_2 C_{22}) \times C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1} (C_{21}\gamma_1 + C_{22}\gamma_2) E_{t-1} Z_{t+i-1},$$

$$P_t = -C_{22}^{-1} C_{21} X_t - C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1} (C_{21}\gamma_1 + C_{22}\gamma_2) E_t Z_{t+i}.$$

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