Speculative Bubbles and Aggregate Boom Bust Cycles

Robert Kollmann (*)
Université Libre de Bruxelles & CEPR

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This paper constructs speculative bubbles in Real Business Cycle models. Building on Blanchard's (1979) classic model of asset price bubbles, the speculative bubbles studied here arise from the absence of a transversality condition (TVC) for production capital. The lack of TVC can be due to an overlapping generations population structure. Speculative bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment, and may occur when there are no shocks to technologies and preferences. It is shown that speculative bubbles can generate bounded boom bust cycles of investment and output. Speculative bubbles are thus a novel potential driver of economic fluctuations.

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(*) Address: Solvay Brussels School of Economics & Management, Université Libre de Bruxelles, CP114, 50 Av. Roosevelt, 1050 Brussels, Belgium
robert_kollmann@yahoo.com www.robertkollmann.com
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1. Introduction

This paper studies speculative bubbles that arise in Real Business Cycle (RBC) models with capital accumulation, when one assumes that there is no transversality condition (TVC) for capital. The absence of the TVC can be due to an overlapping generations (OLG) population structure. The models considered here assume rational expectations. Speculative bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment.

Except for the absence of a TVC, the models analyzed here are entirely standard. The aggregate static equations and Euler equations are *identical* to those of canonical RBC models. If a TVC is imposed, the present models have a unique equilibrium. I show how to construct speculative bubbles that feature *recurrent* boom bust cycles characterized by *bounded* investment and output expansions that are followed by abrupt contractions in real activity. Importantly, recurrent boom bust cycles can arise when there are no shocks to technologies and preferences. The model solutions considered here are globally accurate, and they thus take feasibility and non-negativity constraints for consumption, capital and output into account, even under large deviations from steady state. It is shown that, with speculative bubbles, the unconditional mean of real activity can be close to the no-bubble steady state. Speculative bubbles can generate persistent fluctuations of real activity, and capture key business cycle stylized facts. Speculative bubbles are thus a novel potential driver of economic fluctuations.

The notion of speculative bubbles, defined as multiple equilibria due to the absence of a TVC, was introduced by Blanchard (1979), in a simple linear asset pricing model. This notion has been highly influential in finance, as it provides a powerful narrative about explosive asset market booms that are followed by sudden busts.[1] However, so far, this concept has had much less impact on structural macroeconomics. To the best of my knowledge, the present paper provides the first analysis of Blanchard-type speculative bubbles, in dynamic general equilibrium business cycle models.

Like Blanchard (1979), I assume a speculative bubble process with two states. The economy can either be in a 'boom' state or in a 'bust' (crash) state. In a boom, investment diverges positively from the no-bubble decision rule that holds under the TVC (saddle path). High investment during a boom is sustained by agents' belief that, with positive probability,

[1] See, e.g., Mussa (1990) and Stracca (2004) for references. Google Scholar records 2918 cites (12/2021) for Blanchard (1979) and its companion paper Blanchard and Watson (1982).

investment will grow next period, thereby depressing future consumption and raising the (expected) future marginal utility-weighted return of capital. During a boom, the expansion of investment and output accelerates initially; however, due to decreasing returns, the growth of investment and output ultimately slows down, during a long-lasting boom. An uninterrupted boom has zero probability. At any time, a bust can occur; in a bust, investment drops abruptly, and reverts towards the no-bubble decision rule. Investment busts are triggered by self-fulfilling downward revisions of expected future investment. Transitions between booms and busts are prompted by a random sunspot, and occur with exogenous probabilities. While investment and output "explode" during the early phase of a boom, the global dynamics of investment and output is stable. As investment is high during booms, speculative bubbles exhibit capital over-accumulation -- the average capital stock is higher in bubbly equilibria than in the no-bubble equilibrium (that obtains when the TVC holds).

The analysis here is related to, but fundamentally different from, Ascari et al. (2019) who study *temporary* explosive equilibria, in the textbook linearized three-equation New Keynesian macro model without capital. Under rational expectations, bubbles in that model imply that the expected paths of inflation and output tend to $\pm\infty$. To rule out unbounded paths, the authors postulate *limited* rationality: once an explosive trajectory reaches a threshold, the economy is assumed to revert *permanently* to its unique stable saddle path. Under rational expectations, the future switch to the saddle path would, from the outset, rule out the emergence of bubbles. By contrast, the present paper assumes rational expectations, and capital accumulation is at the heart of the analysis here (however, the paper abstracts from nominal rigidities). In the models considered here, speculative bubbles generate *bounded* output fluctuations.

Explosive (yet bounded) output dynamics in booms distinguishes the model here from the large class of business cycle models with multiple "local" sunspot equilibria whose support is close to steady state. Local sunspot equilibria arise when the number of eigenvalues (of the linearized state-space form) outside the unit circle is less than the number of non-predetermined variables (Blanchard and Kahn (1980); Woodford (1986)). The model ingredients that may give rise to local sunspot equilibria include increasing returns, externalities, financial frictions, certain OLG structures and/or violations of the Taylor principle of monetary policy; see, e.g., Woodford (1988), Schmitt-Grohé (1997), Benhabib and Farmer (1999), Lubik and Schorfheide (2004), Holden (2016, 2021). The sunspot equilibria studied in these models generally satisfy aggregate

TVCs. The model ingredients that deliver local sunspot equilibria may be debatable.[2] None of these ingredients are used in the very simple business cycle models considered in the present paper. In the models studied here, the number of eigenvalues (of the linearized state-space form) outside the unit circle equals the number of non-predetermined variables. The models here have a unique solution when the TVC is imposed (as mentioned above).

Section 2 briefly reviews the speculative asset price bubble analyzed by Blanchard (1979). Sect. 3 constructs speculative bubbles in a simple Long and Plosser (1983) RBC economy, when the TVC is dropped. That model assumes log utility, a Cobb-Douglas production function and full capital depreciation. Exact closed form solutions with speculative bubbles can be derived for that model. Sect. 4 shows how speculative bubbles can be constructed in a richer, more realistic RBC economy with incomplete capital depreciation.

2. Blanchard (1979) speculative asset price bubble

Blanchard (1979) considers a log-linear asset price model of the form $p_t=\beta \cdot E_t p_{t+1}+d_t$, where p_t is the price of a stock (in logs) at date t, while d_t is the (scaled) log dividend. $0<\beta<1$ is the investors' subjective discount factor. Assume, for simplicity that the dividend is constant, and normalized at $d_t=0$, so that $E_t p_{t+1}=\frac{1}{\beta}p_t$. As $\frac{1}{\beta}>1$, the model has a unique non-explosive solution given by $p_t=0\ \forall t$ (Blanchard and Kahn (1980), Prop. 1). Blanchard (1979) pointed out that, if there are no transversality or boundary conditions, the model is also solved by a bubble process $\{p_t\}$ such that $p_{t+1}=0$ obtains with probability π, while $p_{t+1}=[(\frac{1}{\beta})/(1-\pi)]\cdot p_t$ obtains with probability $1-\pi$ $(0<\pi<1)$. If $p_t\neq 0$, then next period the asset price continues to diverge with probability $1-\pi$, while a 'bust' (return to the no-bubble solution $p=0$) occurs with probability π. This bubble process implies that after a bust, non-zero values of p never arise again, i.e. the bubble is 'self-ending'. Recurrent (never-ending) bubbles obtain if the bust implies a price $\Delta\neq 0$: $p_{t+1}=(\frac{1}{\beta}p_t-\Delta\pi)/(1-\pi)$ with probability 1-π and $p_{t+1}=\Delta$ with probability π. Speculative bubbles can exhibit prolonged episodes during which the asset price deviates more and more from its 'fundamental' value (p=0), before abruptly reverting towards that fundamental value.

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[2] E.g., increasing returns/externalities need to be sufficiently strong; in OLG models the steady state interest rate may not exceed the trend growth rate (r≤g) etc. Note that r>g holds in the present model.

3. Speculative bubbles in a Long-Plosser RBC economy without TVC

This Section studies speculative bubbles in a Long and Plosser (1983) RBC economy without TVC. Assume log period utility $u(C_t)=\ln(C_t)$, where $C_t \geq 0$ denotes consumption in period *t*. The production function is:

$$Y_t=\theta_t K_t^{\alpha}\ ,\ 0<\alpha<1\ , \tag{1}$$

where $Y_t, K_t, \theta_t \geq 0$ are output, capital and exogenous total factor productivity (TFP). For simplicity, labor hours are constant and normalized to unity. The resource constraint is

$$C_t+I_t=Y_t,\ \text{with}\ I_t=K_{t+1}. \tag{2}$$

$I_t \geq 0$ is (gross) investment. A 100% capital depreciation rate is assumed, and thus investment equals next period's capital stock. (Sect. 4 considers a model with variable hours and incomplete capital depreciation.) Assume that productivity is bounded. Decreasing returns to capital $(0<\alpha<1)$ then imply that all feasible paths of capital, output and consumption are likewise bounded. The Euler equation for capital is

$$E_t\beta(C_t/C_{t+1})\alpha Y_{t+1}/K_{t+1}=1, \tag{3}$$

where $0<\beta<1$ is the subjective discount factor. Using the resource constraint $K_{t+1}=Y_t-C_t$ one can express the Euler equation (3) as a linear expectational difference equation in the output/consumption ratio $X_t \equiv Y_t/C_t \geq 1$:

$$E_t X_{t+1} = \tfrac{1}{\alpha\beta} X_t - \tfrac{1}{\alpha\beta}. \tag{4}$$

Long and Plosser (1983) assume an *infinitely-lived* representative household. That household's decision problem has a unique solution (as the problem is a well-behaved concave programming problem). The necessary and sufficient optimality conditions of that decision problem are the household's resource constraint and Euler equation (summarized by (4)) and a transversality condition (TVC) that requires that the present discounted value of the capital stock is zero, at infinity:

$$\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=0. \tag{5}$$

Note that $u'(C_{t+\tau})K_{t+\tau+1}=X_{t+\tau}-1$, so the TVC holds iff $\lim_{\tau\to\infty}\beta^{\tau}E_t X_{t+\tau}=0$. A constant output/consumption ratio $X_t=\overline{X}\equiv\frac{1}{1-\alpha\beta}>1\ \ \forall t$ satisfies (4) and the TVC (5). This is the textbook solution of the Long-Plosser model (e.g., Blanchard and Fischer (1989)). Under that solution, consumption and investment are time-invariant shares of output: $C_t = (1-\alpha\beta)Y_t,\ K_{t+1} = \alpha\beta Y_t \quad \forall t$.

In what follows, I postulate that there is no TVC. This gives rise to multiple equilibria. I refer to a process $\{X_t\}$ that satisfies $X_t \geq 1$ and (4), but that differs from the textbook solution (derived under the TVC), as a **speculative bubble equilibrium**, or (speculative) bubble, for short. Speculative bubbles violate the TVC. [3]

The lack of TVC can be justified by the assumption that the economy has an overlapping generations (OLG) population structure. Kollmann (2022) develops an OLG structure with finitely-lived agents in which aggregate (economy-wide) consumption obeys a (quasi-)Euler equation that is isomorphic to the Euler equation of an infinitely-lived representative agent. In that OLG structure, the static equilibrium conditions, and the law of motion of capital can likewise be aggregated across countries; this delivers equations in aggregate variables that are *identical* to corresponding equations for an economy with an infinitely-lived representative agent. When the OLG structure developed by Kollmann (2022) is embedded into a Long-Plosser economy, then equations (1)-(4) continue, thus, to hold. In the OLG structure, each individual agent holds zero assets, at the end of her life, but the path of aggregate assets is not constrained by a terminal condition. Hence, the (infinite-horizon) TVC for *aggregate* capital (5) is *not* an equilibrium condition, in the OLG structure.[4]

Besides assuming an OLG structure, another potential motivation for disregarding the TVC is that detecting TVC violations may be difficult, in economies that are more complicated than the Long-Plosser economy, i.e. in models for which no closed form solution exists (see Sect. 4). TVC violations can be caused by very low-probability events in a distant future. Agents may thus lack the cognitive/computing power to detect TVC violations, so that speculative bubbles can arise (see Blanchard and Watson (1982)).

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[3] (4) implies $E_t X_{t+\tau} = (\frac{1}{\alpha\beta})^\tau (X_t - \overline{X}) + \overline{X}$. Thus, $\lim_{\tau\to\infty} \beta^\tau E_t u'(C_{t+\tau}) K_{t+\tau+1} = \pm\infty$ when $X_t \neq \overline{X}$.

[4] Standard OLG models (e.g. Blanchard, 1985) assume that newborn agents have zero financial wealth at birth (and thus rely on labor income to accumulate wealth). By contrast, Kollmann (2022) assumes that, at each date, newborn agents receive a wealth transfer (from older agents) that is set in such a manner that equilibrium consumption during the first period of life represents a time-invariant share of aggregate consumption. Under complete risk sharing among contemporaneous generations, this ensures that an *aggregate* Euler equation of form (3) holds. The Kollmann (2022) OLG structure thus allows to generate speculative bubbles in tractable models suitable for calibration to quarterly data. By contrast, conventional OLG models are typically much more cumbersome (due to the implied heterogeneity of generations) which makes them less useful for quantitative business cycle analysis.

3.1. Constructing speculative bubbles

(4) is satisfied for any processes

$$X_{t+1}-\overline{X}=\tfrac{1}{\alpha\beta}\cdot(X_t-\overline{X})+\varepsilon_{t+1}, \tag{6}$$

where ε_{t+1} is a random variable whose conditional mean is zero, $E_t\varepsilon_{t+1}=0$. I will focus on bubbles processes that (i) never reach the unit lower bound of the output/consumption ratio and (ii) are not self-ending. When the unit lower bound of the output/consumption ratio is attained, all output is consumed, so that investment and next period's capital stock drop to zero; output, consumption and investment remain at zero forever thereafter. Such an "extinction" equilibrium seems empirically irrelevant. Standard business cycle models concentrate on *recurrent* fluctuations in economic activity. This is why I focus on speculative bubbles that never end. Requirements (i),(ii) necessitate that the output/consumption ratio always strictly *exceeds* the ratio in the no-bubble equilibrium: $X_t>\overline{X}\ \forall t$.[5]

The investment/output ratio is an increasing function of the output/ consumption ratio: $K_{t+1}/Y_t=(Y_t-C_t)/Y_t=1-1/X_t$. Thus, $X_t>\overline{X}$ implies that the investment/output ratio is larger in the recurrent speculative bubbles studied here than in the text-book no-bubble equilibrium. Hence, the bubbles considered here exhibit capital over-accumulation. Note also that $X_t>\overline{X}$ implies that the *expected* path of the output/consumption ratio explodes (from (6)): $\lim_{s\to\infty}E_tX_{t+s}=\infty$. However, consumption, capital and output are *bounded,* as any feasible path for these variables is bounded, due to decreasing returns to capital (and given bounded TFP); see above.

In what follows, I construct recurrent speculative bubbles such that $X_t\geq\overline{X}+\Delta\ \forall t$ holds for a small constant $\Delta>0$. By analogy to the recurrent (never-ending) Blanchard (1979) asset-price bubble, I consider a two-state bubble process for the output/consumption ratio $\{X_t\}_{t\geq0}$ defined by:

(i) $X_0\geq\overline{X}+\Delta$;

(ii) $X_{t+1}=X_{t+1}^L\equiv\overline{X}+\Delta$ with probability $0<\pi<1$,

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[5] When $X_t<\overline{X}$, then the economy hits the unit lower bound of the output/consumption ratio almost surely in a later period, as the autoregressive coefficient $1/(\alpha\beta)$ of (6) exceeds unity. When $X_t=\overline{X}$, then $X_\tau=\overline{X}\ \forall\tau\geq t$ is the only process going forward that never hits the unit lower bound.

and $X_{t+1}=X_{t+1}^{H}\equiv\overline{X}+\{\frac{1}{\alpha\beta}[X_t-\overline{X}]-\Delta\pi\}/(1-\pi)\equiv\Psi(X_t)$ with probability 1-π for $t\geq 0$. (7)

Whether X_{t+1}^{L} or X_{t+1}^{H} is realized depends on an exogenous i.i.d. sunspot (the probability π is exogenous). Note that $X_{t+1}^{L}<X_{t+1}^{H}$. Thus, states X_{t+1}^{L} and X_{t+1}^{H} will be referred to as investment busts and booms, respectively.

Given the sequence $\{X_t\}_{t\geq 0}$, the path of capital $\{K_{t+1}\}_{t\geq 0}$ can be generated recursively (for an exogenous initial capital stock K_0) using $K_{t+1}=\{1-1/X_t\}\theta_t(K_t)^{\alpha}$ for $t\geq 0$.

An *uninterrupted* infinite sequence of investment booms (X^{H}) would drive the output/consumption ratio to infinity and the investment/output ratio to unity, while capital (and output) would converge towards its (finite) upper bound. Of course, an uninterrupted investment boom run has zero probability. At any time, the output/consumption ratio can drop to $\overline{X}+\Delta$, with probability π. This ensures that the investment/output ratio undergoes recurrent fluctuations. If the bust probability π is sufficiently big and if Δ is close to zero, then speculative bubbles induce fluctuations of real activity that remain most of the time near the steady state of the no-bubble economy. This is the case in the stochastic simulations reported below.

What expectations sustain the speculative bubble? In equilibrium, agents expect at date t that X_{t+1} will equal $X_{t+1}^{L}=\overline{X}+\Delta$ or $X_{t+1}^{H}=\Psi(X_t)$ with probabilities π and 1-π, respectively (the function Ψ is defined in (7)). Note that X_{t+1}^{L} and X_{t+1}^{H} are known at t. At t+1, agents are free to select a value of X_{t+1} that differs from X_{t+1}^{L} or X_{t+1}^{H}; however, in equilibrium, they chose not to do so because a choice $X_{t+1}\in\{X_{t+1}^{L},X_{t+1}^{H}\}$ is 'validated' by their date t+1 expectations about X_{t+2}. Assume that an investment **bust** occurs in t+1, so that agents choose $X_{t+1}=\overline{X}+\Delta$; in equilibrium, this choice is sustained by agents' expectation (at t+1) that X_{t+2} will equal $\overline{X}+\Delta$ or $\Psi(\overline{X}+\Delta)$ with probabilities π and 1-π, respectively. By contrast, if an investment **boom** occurs at t+1, then agents choose $X_{t+1}=X_{t+1}^{H}\equiv\Psi(X_t)$; this choice is supported by the expectation (at t+1) that X_{t+2} will equal $\overline{X}+\Delta$ or $\Psi(X_{t+1}^{H})=\Psi(\Psi(X_t))$ with probabilities π and 1-π, respectively. Note that $X_t\geq\overline{X}+\Delta$ implies that $\Psi(\overline{X}+\Delta)<\Psi(\Psi(X_t))$ holds. This shows that, in an investment boom (at t+1), agents expect a higher future investment/output ratio than in an investment bust (at t+1). Booms and busts reflect hence self-fulfilling variations in agents' expectations about the future state of the economy. An investment boom [bust] is triggered by a more [less] optimistic assessment of next period's investment/output ratio. High investment during a boom is sustained by agents' belief that, with positive probability, investment will continue to grow next period, thereby depressing future consumption and raising the (expected) future marginal utility-weighted return of capital.

3.2. Quantitative results

I next discuss stochastic model simulations. I set α=1/3 and β=0.99, as is standard in quarterly macro models. To assess whether speculative bubbles alone can generate a realistic business cycle, I assume that TFP is constant. The bust probability is set at $\pi=0.5$. I set $\Delta=3.8\times10^{-6}$ as for that value the bubble model matches the standard deviation of Hodrick-Prescott (HP) filtered historical US real GDP (see below). [6]

Figure 1 shows representative simulated paths of output (Y, continuous black line), consumption (C, red dashed line) and investment (I, blue dash-dotted line).[7] The bubble model generates sudden, but short-lived, expansions in output and investment. In a boom, the rapid rise in investment is accompanied by a contraction in consumption. At any time, a bust can occur; in a bust, investment drops abruptly.

Table 1 (Row (a)) reports model-generated standard deviations (in %) and cross-correlations of HP filtered (logged) output, consumption and investment; also shown are mean values of these variables. All model-generated business cycle statistics reported in Table 1 (and Table 2 discussed below) are based on one simulation run of T=10000 periods. The reported theoretical business cycle statistics (standard deviations, correlations) are median statistics computed across rolling windows of 200 periods.[8] Mean values (of Y,C and I) are computed using the whole simulation run (T periods) and expressed as % deviations from the no-bubble steady state.

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[6] In the simulations, the law of motion of the bubbly output/consumption ratio (7) is initiated with a ratio $X_0=\overline{X}+\Delta$. I set the initial capital stock K_0 at the no-bubble steady state capital stock. The effect of initial values on subsequent simulated values vanishes fast and does not noticeably affect moments over a long simulation run.

[7] The depicted simulated paths of Y,C,I are normalized by steady state no-bubble output.

[8] The HP filter is applied separately in each respective time window. 200-periods windows are used, as the historical business cycle statistics shown in Table 1 pertain to a sample of 200 quarters (see below).

To evaluate the model predictions, Table 1 also reports US historical business statistics based on HP filtered quarterly data (logged) for the period 1968-2017 (see Row (b)). The empirical standard deviations of GDP, consumption and investment are 1.47%, 1.19% and 4.96%, respectively. In the data, consumption and investment are strongly procyclical; these variables and GDP are highly serially correlated.

The model-predicted standard deviations of output, consumption and investment are 1.47%, 3.39% and 4.42%, respectively (see Row (a) of Table 1). Thus, consumption is more volatile in the model than in the data, but the model matches well the high empirical volatility of investment. In the bubble economy, consumption and investment are procyclical; output and investment are predicted to be positively serially correlated, while consumption is predicted to be negatively autocorrelated. Average output and investment are 0.5% and 2.1% higher than in the no-bubble steady state, while consumption is 0.3% lower. Thus, the mean of these endogenous variables is close to the no-bubble steady state.

Capital over-accumulation (compared to the no-bubble equilibrium) implies that the bubble economy is 'dynamically inefficient'. Abel et al. (1989) propose an empirical test of dynamic efficiency. Their key insight is that, in a dynamically efficient economy, income generated by capital (i.e. output minus the wage bill) exceeds investment. Abel et al. (1989, Table 1) show that, in US data 1929-1985, this condition is met in all years of their sample. The US historical sample average of the (capital income-investment)/GNP ratio is 13.41%.[9]

In the bubbly Long-Plosser economy, the (capital income – investment)/GDP ratio is positive in 97.01% of all quarters, but the average ratio is slightly negative, -0.09%. Note that, in the no-bubble version of the Long-Plosser economy, the (capital income – investment)/GDP ratio equals $\alpha(1-\beta)=0.33\%$, which is only slightly greater than zero, and much smaller than the empirical ratio. Thus, even modest dynamic inefficiency produces a negative mean capital income – investment gap. As shown in the next Section, an RBC economy with incomplete capital depreciation can generate speculative bubbles with sizable positive mean capital income – investment gaps.

[9] The (capital income-investment)/GNP ratio is likewise positive and sizable in post-1985 data. Mean ratio 1968-2017 (sample period used for business cycle moments reported in Table 1): 17.58%.

4. Speculative bubbles in an RBC economy (no TVC) with incomplete capital depreciation

I next show how speculative bubbles can be constructed in a richer, more realistic RBC economy with incomplete capital depreciation (and variable labor). As before, I postulate that there is no TVC for capital.

The period utility function is now assumed to be $u(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t),\ \Psi>0,$ where $0\leq L_t\leq 1$ are hours worked. The household's total time endowment (per period) is normalized to one, so $1-L_t$ is leisure.[10] The resource constraint and the output technology are

$$C_t+K_{t+1}=Y_t+(1-\delta)K_t \text{ with } Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \tag{8}$$

where $0<\delta<1$ is the capital depreciation rate. θ_t (TFP) is exogenous and follows the bounded AR(1) process $\ln(\theta_{t+1})=\rho\ln(\theta_t)+\varepsilon^{\theta}_{t+1},\ 0\leq\rho<1,$ where $\varepsilon^{\theta}_{t+1}$ is a white noise that equals $-\sigma_\theta$ or σ_θ with probability ½ $(\sigma_\theta\geq 0)$. The standard deviation of the $\varepsilon^{\theta}_{t+1}$ is thus σ_θ.[11] The economy has these efficiency conditions

$$C_t\Psi/(1-L_t)=(1-\alpha)\theta_t(K_t)^{\alpha}(L_t)^{-\alpha} \text{ and} \tag{9}$$

$$E_t\beta\{C_t/C_{t+1}\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(L_{t+1})^{1-\alpha}+1-\delta)=1. \tag{10}$$

(9) indicates that the household's marginal rate of substitution between leisure and consumption is equated to the marginal product of labor, while (10) is the date *t* Euler equation for capital.

(8) and (9) pin down consumption and hours worked as functions of K_{t+1},K_t,θ_t:

$$C_t=\gamma(K_{t+1},K_t,\theta_t) \text{ and } L_t=\eta(K_{t+1},K_t,\theta_t). \tag{11}$$

Substituting these expressions into the Euler equation (10) gives:

$$E_tH(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)=1, \text{ where} \tag{12}$$

$$H(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)\equiv\beta\{\gamma(K_{t+1},K_t,\theta_t)/\gamma(K_{t+2},K_{t+1},\theta_{t+1})\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(\eta(K_{t+2},K_{t+1},\theta_{t+1}))^{1-\alpha}+1-\delta).$$

The model thus boils down to an expectational difference equation in capital. The conventional no-bubble model solution, that obtains when the TVC for capital (5) is imposed, is

[10] The upper bound on labor hours implies that capital and output are bounded. Some widely used preference specifications (e.g., $u(C_t,L_t)=\ln(C_t)-\Psi\cdot(L_t)^{\mu},\ L_t\geq 0,\ \mu>1$) do not impose an upper bound on labor. Then speculative bubbles may induce unbounded growth of hours, capital and output.

[11] The discrete distribution of the TFP innovation simplifies the computation of conditional expectations in the numerical model solution.

described by a unique decision rule $K_{t+1}=\lambda(K_t,\theta_t)$ (e.g., Schmitt-Grohé and Uribe (2004)). A speculative bubble is a process $\{K_t\}$ that satisfies the Euler equation (12) but that violates the TVC and thus deviates from the no-bubble decision rule.

Recurrent speculative bubbles

By analogy to the bubble process in the Long-Plosser economy without TVC (Sect. 3), I consider bubbles in which, given date *t* information, the capital stock K_{t+1} takes one of two values: $K_{t+1}\in\{K_{t+1}^L,K_{t+1}^H\}$ with exogenous probabilities π and $1-\pi$, respectively ($0<\pi<1$), where $K_{t+1}^L\equiv\lambda(K_t,\theta_t)e^{\Delta}$, for a small constant $\Delta>0$. With probability π, the capital stock thus takes a value close to the no-bubble decision rule (as in the bubbly Long-Plosser economy). At date *t*, an exogenous i.i.d. sunspot (independent of TFP or any past endogenous variables) determines whether K_{t+1}^L or K_{t+1}^H is realized. At *t*, agents anticipate that K_{t+2} too takes one of two values $K_{t+2}\in\{K_{t+2}^L,K_{t+2}^H\}$ with probabilities π and $1-\pi$, respectively, with $K_{t+2}^L=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$. The date *t* Euler equation (12) can thus be written as:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)+(1-\pi)\cdot E_t H(K_{t+2}^H,K_{t+1},K_t,\theta_{t+1},\theta_t)=1 \text{ for } K_{t+1}\in\{K_{t+1}^L,K_{t+1}^H\}. \quad (13)$$

The numerical simulations below consider equilibria in which, conditional on date *t* information, a TFP innovation at *t*+1 has an equiproportional effect on K_{t+2}^L and on K_{t+2}^H. Specifically, I construct equilibria in which $K_{t+2}^H=s_t^H\cdot K_{t+2}^L$ holds, where $s_t^H>0$ is known at *t*. Solving (13) for s_t^H (at each date) then pins down the equilibrium capital process. Computational details can be found in the Appendix (for online publication).

I set $\Delta>0$, as this is needed to generate *recurrent* bubbles (With $\Delta\leq 0$, bubbles are self-ending or ultimately hit the zero capital corner, as in the Long-Plosser economy without TVC.)

As in the bubbly Long-Plosser economy, the dynamics of capital reflects self-fulfilling variations in agents' expectations about *future* capital. Due to decreasing returns to capital and bounded TFP, the paths of capital and output are bounded. An *uninterrupted* sequence of investment booms (an infinite string of K^H realizations) would drive the capital stock towards its upper bound. However, an uninterrupted boom has zero probability. At any time, the capital stock can revert towards the no-bubble decision rule, with probability π. For values of Δ close to zero, and a sufficiently high bust probability π (as assumed in the simulations below), capital and output remain close to the range of the no-bubble equilibrium, most of the time, and the mean of capital and output is close to the no-bubble steady state.

4.1. Quantitative results

I again set $\alpha=1/3, \beta=0.99$. The capital depreciation rate is set at $\delta=0.025$. The preference parameter Ψ (utility weight on leisure) is set so that the Frisch labor supply elasticity is unity, at the no-bubble steady state.[12] Parameters in this range are conventional in quarterly macro models (e.g., King and Rebelo (1999)). I set the autocorrelation of TFP at $\rho=0.979$, while the standard deviation of TFP innovations is set at $\sigma_\theta=0.72\%$, as suggested by King and Rebelo (1999). The numerical simulations assume $\Delta=10^{-6}$ for all variants of the bubble model. That value generates standard deviations of real activity that are roughly in line with empirical statistics. I report results for two values of the bust probability: $\pi=0.2$ and $\pi=0.5$.

Table 2 reports simulated business cycle statistics (of HP filtered logged variables) for several model variants (see Cols. (1)-(10)), as well as historical US business cycle statistics (Col. (11)). Standard deviations (in %) of output (Y), consumption (C), investment (I) and hours worked (L) are shown, as well as correlations of these variables with output, autocorrelations and mean values. Also reported are means of Y,C,I,L and the mean of the (capital income-investment)/GDP ratio, as well as the fraction of periods in which this ratio is positive.

Cols. (1)-(4) of Table 2 pertain to bubble model variants with just bubble (sunspot) shocks (constant TFP assumed). Cols. (5)-(8) consider bubble model variants with joint bubble and TFP shocks. Cols. (9),(10) assume a no-bubble model (TVC imposed) with TFP shocks.[13] Cols. (1),(3),(5),(7) assume a bust probability $\pi=0.5$, while Cols. (2),(4),(6),(8) assume $\pi=0.2$. Cols. labelled 'Unit Risk Aversion' ('Unit RA') assume log utility, $u(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t)$. Columns labelled 'High RA' assume $u(C_t,L_t)=\ln(C_t-\overline{C})+\Psi\cdot\ln(1-L_t)$, where $\overline{C}$ is a constant that is set at 0.8 times no-bubble steady state consumption. In the 'High RA' case, consumption thus

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[12] (9) implies that the Frisch labor supply elasticity (LSE) with respect to the real wage (marginal product of labor) is $LSE=(1-L)/L$ at the steady state, where L are steady state hours worked. Ψ is set such that $L=0.5$, as then $LSE=1$.

[13] The no-bubble model is solved using a second-order Taylor approximation, as it is well-know that this approximation is very accurate for standard (no-bubble) RBC models (e.g., Kollmann et al. (2011a,b)).

has a strictly positive lower bound, $C_t \geq \overline{C} > 0$; the coefficient of relative risk aversion is 5, at steady state consumption.

Col. (1) of Table 2 assumes a variant of the bubble model with unit risk aversion and a bust probability π=0.5; fluctuations are just driven by bubble shocks. The predicted standard deviations of output, consumption, investment and hours worked are 0.49%, 1.08%, 4.29% and 0.74%, respectively. In line with the historical data, investment is predicted to be more volatile than output. However, the model (with unit risk aversion) predicts that consumption is more volatile than output, which is counterfactual. In the model, consumption is negatively correlated with output (a positive bubble shock raises investment; this crowds out consumption, which raises labor supply and thereby boosts output).[14] However, the model predicts that investment and hours worked are strongly procyclical, as is consistent with the data. In the model, output, consumption, investment and hours worked are positively serially correlated, but predicted autocorrelations (about 0.35) are smaller than the empirical autocorrelations (about 0.9).

A lower bust probability π=0.2 generates more persistent booms in real activity. As shown in Col. 2, for the bubble model variant with unit risk aversion, the autocorrelation of real activity is about 0.6 for π=0.2; however, consumption remains more volatile than output. Model variants with 'High Risk Aversion (RA)' generate less consumption volatility—those variants capture the fact that consumption is less volatile than output; see Cols. (3) and (4) of Table 2, where π=0.5 and π=0.2 are assumed.

In summary, the bubble model with constant TFP can generate persistent fluctuations, as well as a realistic volatility of output and aggregate demand components.

The no-bubble model driven by stochastic TFP shocks underpredicts the volatility of real activity, but it captures the fact that consumption is less volatile than output, while investment is more volatile (see Table 2, Cols. (9) and (10)). In the no-bubble model, consumption and investment are pro-cyclical; furthermore, real activity is highly serially correlated

The bubble economy with joint bubble shocks and TFP shocks generates fluctuations in real activity that are more volatile than the fluctuations exhibited by the no-bubble economy (see Table 2, Cols. (5)-(8)). The bubble equilibrium with TFP shocks is thus closer to the historical amplitude of business cycles.

[14] This is a familiar feature of flex-wage models driven by investment shocks; e.g., Coeurdacier et al. (2011).

Figure 2 shows simulated paths of output (Y, continuous black line), consumption (C, red dashed line), investment (I, dark blue dash-dotted line) and hours worked (L, light blue dotted line), for the model version with 'High Risk Aversion' and a bust probability π=0.2. Panels (1) and (2) of Fig.2 show results for the bubble economy with just bubble shocks, and for the bubble economy with joint bubble and TFP shocks, respectively. Panel (3) pertains to a no-bubble economy with TFP shocks.[15]

We see that bubble shocks induce relatively widely spaced output and investment booms (see Panel (1)). In most periods, output, consumption, investment and output remain close to the no-bubble steady state. Panels (2) and (3) of Fig. 2 show that the effect of bubbles on simulated series is clearly noticeable: the bubble economy with joint bubble and TFP shocks exhibits more rapid, short-lived, increases in investment, labor hours and output, that are followed by sharper contractions, than the no-bubble economy with TFP shocks.

In the bubble economies considered in Table 2, the mean of output, consumption and investment is again close to the no-bubble steady state (as in the bubbly Long-Plosser economy studied in Sect. 3).[16] For all bubble model variants in Table 2, the average (capital income – investment)/GDP ratio is positive and large (unlike in the Long-Plosser model); the average ratio ranges between 8.5% and 9.2%, and it is only slightly smaller than the value of that ratio in the no-bubble steady state, 9.59%.[17] Capital income exceeds investment in close to 100% of all periods. This highlights the difficulty of detecting violations of the TVC (dynamic inefficiency), as discussed above.

5. Speculative bubbles in two-country RBC models

Kollmann (2020) constructs speculative bubbles that arise in two-country RBC economies of the Dellas (1986) and Backus et al. (1994) type, when there are no transversality conditions for domestic and foreign capital. It is shown that, with integrated global financial markets,

[15] The Y, C and I series plotted in Fig. 2 are normalized by no-bubble steady state output; hours worked (L) are normalized by steady state hours. The same sequence of sunspots are used in Panels (1) and (2); the same TFP shocks are used in Panels (2) and (3). Simulated paths for the other model variants considered in Table 2 are shown in Kollmann (2020).

[16] In Table 2, mean values of Y,C,I,L are reported as % deviations from the no-bubble steady state. The mean (capital income – investment)/GDP ratio is *not* expressed as a % deviation from steady state.

[17] The no-bubble steady state (capital income – investment)/GDP ratio is $\alpha r/(\delta+r)$ with $r \equiv (1-\beta)/\beta$.

speculative bubbles must be perfectly correlated across countries. Global bubbles may, thus, help to explain the synchronization of international business cycles.

6. Conclusion

This paper studies speculative bubbles that arise in Real Business Cycle models, when one assumes that there is no transversality condition (TVC) for capital. The absence of the TVC can be due to an overlapping generations population structure. Speculative bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment, and can occur when there are no shocks to technologies and preferences. It is shown that speculative bubbles can generate bounded boom bust cycles of investment and output. Speculative bubbles are thus a novel potential driver of economic fluctuations.

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Table 1. Long-Plosser economy with speculative bubbles: business cycle statistics

	Standard dev. (%)			Corr. with Y		Autocorrelations			Mean [% deviation from SS]		
	Y	*C*	*I*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
(a) Predicted business cycle statistics											
	1.47	3.39	4.42	0.31	0.45	0.44	-0.17	0.44	0.49	-0.32	2.15
(b) Historical business cycle statistics											
	1.47	1.19	4.96	0.87	0.92	0.87	0.89	0.92			

Notes: Row (a) reports simulated business cycle statistics for a Long-Plosser economy (full capital depreciation) with speculative bubbles (no transversality condition); see Sect. 3 of paper. *Y*: output; *C*: consumption; *I*: investment.

In the simulated model, fluctuations are just driven by bubble shocks (constant TFP assumed). Bust probability π=0.5.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods. The reported simulated standard deviations, correlations with output and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Simulated series were logged and HP filtered (the HP filter was applied separately to each window of 200 periods). 'Means' (of *Y*,*C*,*I*) are sample averages over the total sample of T periods; means are expressed as % deviations from the steady state of the no-bubble economy.

Row (b) reports US historical business cycle statistics (quarterly data), 1968q1-2017q4. The empirical data are taken from BEA NIPA (Table 1.1.3). *Y*: GDP; *C*: personal consumption expenditures; *I*: fixed investment.

Table 2. RBC economy with incomplete capital depreciation: business cycle statistics

	Bubble model (no TVC)								*No-bubble model*		
	Bubble shocks; no TFP shocks				Bubble & TFP shocks				TFP shocks		
	Unit Risk aversion		High RA		Unit RA		High RA				
	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.27	1.60	0.98	1.57	1.14	0.72	1.47
C	1.08	2.63	0.29	0.61	1.16	2.71	0.38	0.72	0.49	0.26	1.19
I	4.29	9.38	3.22	6.51	5.38	9.85	3.86	6.72	3.33	2.20	4.96
L	0.74	1.73	1.04	2.18	0.82	1.70	1.05	2.22	0.34	0.30	1.06
Correlations with output											
C	-0.97	-0.95	-0.99	-0.98	0.04	-0.54	0.01	-0.62	0.95	0.99	0.87
I	0.98	0.96	0.99	0.99	0.89	0.86	0.97	0.98	0.99	0.99	0.92
L	0.99	0.97	0.99	0.99	0.79	0.81	0.45	0.82	0.98	-0.96	0.82
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.65	0.68	0.57	0.66	0.71	0.70	0.87
C	0.33	0.60	0.35	0.62	0.43	0.62	0.53	0.65	0.76	0.72	0.89
I	0.36	0.63	0.37	0.64	0.53	0.65	0.51	0.65	0.70	0.70	0.92
L	0.34	0.61	0.35	0.62	0.45	0.62	0.41	0.63	0.70	0.74	0.92
Means [% deviation from no-bubble steady state]											
Y	1.41	2.80	1.25	2.12	1.37	2.75	1.31	2.17	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.68	1.34	0.33	0.55	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.61	7.28	4.44	7.40	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.34	0.73	0.01	-0.03	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.16	8.78	8.92	8.53	9.58	9.58	13.41
Fraction of periods for which capital income > investment [in %]											
	99.20	96.31	99.55	97.72	99.20	96.43	99.37	97.74	100	100	100

Notes: This Table reports simulated business cycle statistics for an RBC economy with incomplete capital depreciation; see Sect. 4 of paper. *Y*: output; *C*: consumption ; *I*: investment; *L*: hours worked.

Cols. (1)-(4) pertain to bubble model variants (no transversality condition, TVC) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Cols. (5)-(8) pertain to bubble model variants, driven by joint bubble and TFP shocks. Cols. (9),(10) pertain to a no-bubble model (TVC imposed) driven by TFP shocks. 'Unit Risk Aversion': log utility; 'High Risk Aversion (RA)': consumption utility given by $\ln(C_t-\overline{C})$, with $\overline{C}>0$. π : bust probability of bubble process.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations with output and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods for which capital income > investment' likewise pertains to the whole simulation run of T periods.

Col. (11) reports US historical statistics (quarterly data). Statistics for *Y*,*C*,*I*: see Table 1. The empirical measure for '*L*' is: 'Total Employment' (Source: CPS, as reported by FRED database, series CE160V). Historical mean of (capital income – investment)/GDP ratio: based on US annual data 1929-1985 reported by Abel et al. (1989)).

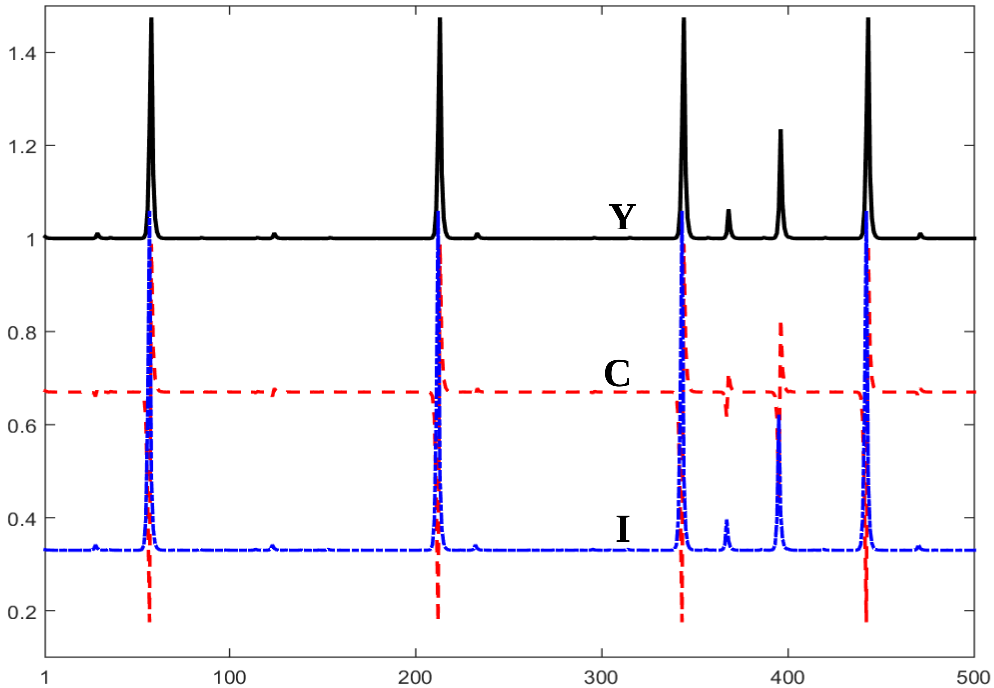

Figure 1. Long & Plosser economy with bubbles (no transversality condition): simulated paths
Simulated paths of output (Y, continuous black line), consumption (C, red dashed line) and investment (I, blue dash-dotted line) are normalized by no-bubble steady state output.

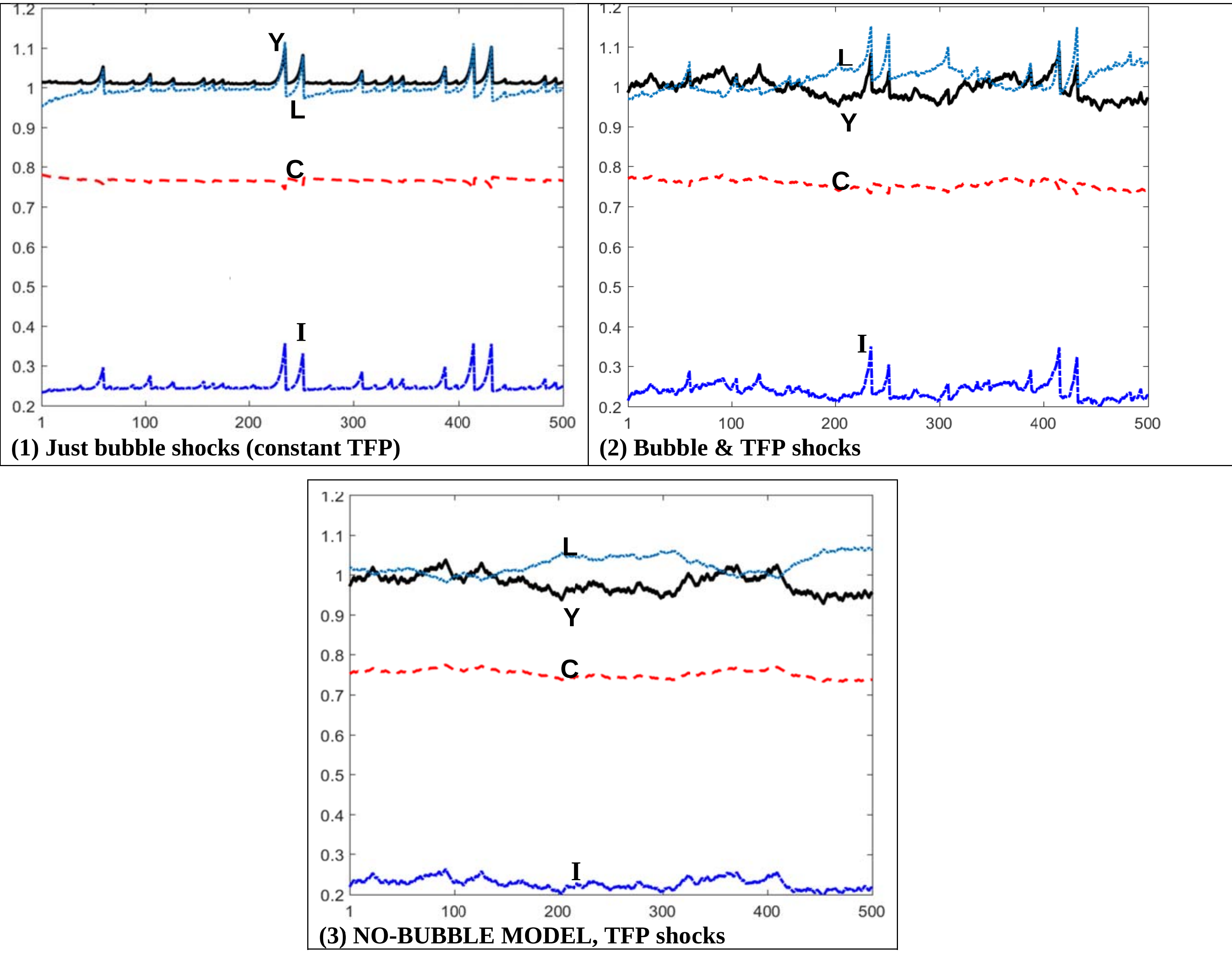

Figure 2. RBC economy with incomplete capital depreciation: simulated paths

This Figure assumes the RBC economy with incomplete capital depreciation, 'High risk aversion' and a bust probability π=0.2 described in Sect. 4. Panel (1) pertains to a bubble model (no transversality condition) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Panel (2) pertains to a bubble model driven by joint bubble and TFP shocks. Panel (3) pertains to a no-bubble model, driven by TFP shocks. Simulated paths of output (Y, continuous black line), consumption (C, red dashed line), investment (I, dark blue dash-dotted line) and hours worked (L, light blue dotted line) are shown. Y, C and I: normalized by no-bubble steady state output. L: normalized by steady state hours.

Y C I L

FOR ONLINE PUBLICATION

APPENDIX

Further discussion of the bubbly RBC economy (no TVC) with incomplete capital depreciation (Sect. 4)

The numerical simulations of the model developed in Sect. 4 consider bubbles for which, conditional on date t information, a TFP innovation at t+1 has an equiproportional effect on $K^L_{t+2}=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$ and on K^H_{t+2}. Specifically, I consider equilibria in which $K^H_{t+2}=s^H_t\cdot K^L_{t+2}$ holds, where $s^H_t>0$ is in the date t information set. Thus, $K^H_{t+2}=s^H_t\cdot\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$.[18] This greatly simplifies the bubble model's solution. Substituting the formula for K^H_{t+2} into the Euler equation (13) gives:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)+(1-\pi)\cdot E_t H(s^H_t\cdot\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)=1, \qquad \text{(A.1)}$$

for $K_{t+1}\in\{K^L_{t+1},K^H_{t+1}\}$, with $K^L_{t+1}=\lambda(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}=s^H_{t-1}\cdot K^L_{t+1}$. For given values of K_t, K_{t+1}, θ_t the Euler equation (A.1) only involves one unknown endogenous variables: s^H_t. Solving (A.1) for s^H_t at each date pins down the equilibrium capital process (see further discussion below). Given the equilibrium capital process, on can compute consumption, hours and output using (11).

As explained in the main text, I set Δ>0, because a strictly positive Δ is needed to generate *recurrent* bubbles. As in the Long-Plosser economy without TVC, bubbles are self-ending when Δ=0; by contrast, Δ<0 implies that the capital stock ultimately reaches zero.[19]

Computational aspects

I. Solving for consumption and labor hours using the static equations

The static model equations can be used to solve for consumption and labor hours as functions of capital and TFP (see (11) in main text). Note that the labor supply equation (9) can be written as

$$C_t=[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}(L_t)^{-\alpha}(1-L_t). \qquad \text{(A.2)}$$

The date t resource constraint of the economy is $C_t+K_{t+1}=Y_t+(1-\delta)K_t$, where $Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}$. Substituting (A.2) into the resource constraint gives:

$$[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}(L_t)^{-\alpha}(1-L_t)=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}+(1-\delta)K_t-K_{t+1}.$$

[18] The AR(1) specification of TFP implies $\theta_{t+1}=(\theta_t)^{\rho}\cdot\exp(\varepsilon^{\theta}_{t+1})$, where $\varepsilon^{\theta}_{t+1}$ is the TFP innovation at t+1. The chosen specification of K^L_{t+2}, K^H_{t+2} implies that $\partial\ln(K^H_{t+2})/\partial\varepsilon^{\theta}_{t+1}=\partial\ln(K^L_{t+2})/\partial\varepsilon^{\theta}_{t+1}$; thus, an unexpected change in date t+1 productivity affects K^H_{t+2} and K^L_{t+2} by the same (relative) amount.

[19] Consider the dynamics that obtains when Δ=0. Assume that, at date t, the sunspot selects (with Δ=0) $K_{t+1}=K^L_{t+1}\equiv\lambda(K_t,\theta_t)$. Then Euler equation (A.1) is solved by $s^H_t=1$, so that $K^H_{t+2}=\lambda(K_{t+1},\theta_{t+1})$. This follows from the fact that $E_t H(\lambda(\lambda(K_t,\theta_t),\theta_{t+1}),\lambda(K_t,\theta_t),K_t)=1$ (Schmitt-Grohé and Uribe (2004), eqn. (4)). Thus $K_{t+2}=K^H_{t+2}=K^L_{t+2}=\lambda(K_{t+1},\theta_{t+1})$ and $K_{t+s+1}=\lambda(K_{t+s},\theta_{t+s})$ also holds ∀ s>1. In all subsequent periods the dynamics of the capital stocks is hence governed by the no-bubble decision rule, i.e. the bubble has ended.

Equivalently: $1=A_{1,t}\cdot(L_t)^{\alpha}+A_2 \cdot L_t$, with $A_{1,t}\equiv-[K_{t+1}-(1-\delta)K_t]/\{[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}\}$, $A_2\equiv[1+\Psi/(1-\alpha)]$. For the assumed capital elasticity of output α=1/3, this (cubic) equation has a unique closed form solution for date t hours worked L_t as a function of K_{t+1},K_t,θ_t. Substitution of the formula for hours into (A.2) gives a closed form formula for consumption C_t (see (11)).

II. Euler equation

TFP is assumed to follow the AR(1) process $\ln(\theta_{t+1})=\rho\ln(\theta_t)+\varepsilon^{\theta}_{t+1}$, $0\leq\rho<1$, where $\varepsilon^{\theta}_{t+1}$ is a discrete innovation that equals $\varepsilon^{\theta}_{t+1}=-\sigma_\theta$ or $\varepsilon^{\theta}_{t+1}=\sigma_\theta$ with probability 1/2, respectively, where $\sigma_\theta\geq 0$. The Euler equation (A.1) can, thus, be written as:

$$\begin{aligned}&\pi\{\tfrac{1}{2}H(\lambda(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(\lambda(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}\\ &(1-\pi)\{\tfrac{1}{2}H(s^H_t\cdot\lambda(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(s^H_t\cdot\lambda(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}=1\end{aligned} \tag{A.3}$$

for $K_{t+1}\in\{K^L_{t+1};K^H_{t+1}\}$, where $K^L_{t+1}=\lambda(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}=s^H_{t-1}K^L_{t+1}$.

In the numerical simulations, I approximate the no-bubble decision rule λ using a second-order (log-quadratic) Taylor expansion. Let $\hat{\lambda}(K_t,\theta_t)$ be the second-order Taylor expansion of the no-bubble decision rule λ. In the numerical simulations, I thus define K^L_{t+1} as $K^L_{t+1}\equiv\hat{\lambda}(K_t,\theta_t)$ $\forall t$. The simulations are hence based on a version of Euler equation (A.3) in which λ is replaced by $\hat{\lambda}$:

$$\begin{aligned}&\pi\{\tfrac{1}{2}H(\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}\\ &(1-\pi)\{\tfrac{1}{2}H(s^H_t\cdot\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(s^H_t\cdot\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}=1\end{aligned} \tag{A.4}$$

for $K_{t+1}\in\{K^L_{t+1};K^H_{t+1}\}$, where $K^L_{t+1}=\hat{\lambda}(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}=s^H_{t-1}K^L_{t+1}$.

Conditional on K_t,K_{t+1},θ_t, this equation can be used to determine s^H_t. I employ a bisection method for that purpose.

Like the value of the bust probability π, the specification of the bust capital stock K^L is not tied down by economic theory. The only restriction is that the resulting law of motion for capital has to be bounded and strictly positive. I verified that the bubble equilibrium constructed using $\hat{\lambda}$ meets this criterion. For model variants with constant TFP, I also computed the no-bubble decision rule $K_{t+1}=\lambda(K_t,\theta)$ using a shooting algorithm (Judd (1998), ch.10). The second-order approximation and the shooting algorithm give no-bubble decision rules that are very close, even when capital K_t is far from the no-bubble steady state. The resulting speculative bubbles too are very similar. Computing $\hat{\lambda}$ is much faster.

III. Initial capital and equilibrium recursion

The stochastic simulations of the bubble economy start at an initial date t=0. The exogenous initial capital stock K_0 is assumed to equal the no-bubble steady state capital stock. K_1 (the period 1 capital stock set at t=0) is indeterminate, in the bubble economy. I assume

$K_1 = \lambda(K_0, \theta_0)e^{\Delta}$. The following recursion allows to simulate an equilibrium path of capital in periods $t>1$, for an exogenous path of TFP $\{\theta_t\}_{t\geq 0}$:

● Given K_0, K_1 and θ_0, the date t=0 Euler equation (A.4) determines s_0^H.

● At t=1, realized productivity θ_1 pins down the two possible values of the date t+2 capital stock: $K_2^H = s_0^H \cdot K_2^L$ where $K_2^L = \lambda(K_1, \theta_1)e^{\Delta}$. An exogenous random draw (sunspot) then determines at t=1 whether K_2 equals K_2^L (probability π) or K_2^H (probability 1-π). Given K_1, K_2, θ_1 the t=1 Euler equation (A.4) pins down s_1^H.

● The same process is repeated in all subsequent periods.

The effect of K_0, K_1 on endogenous variables in subsequent periods vanishes as time progresses; initial conditions do not affect moments over a long simulation run.

References

Judd, Kenneth, 1998. Numerical Methods in Economics. Cambridge, MA: MIT Press.